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Mechanics of continuous media

A variational principle

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The investigation of the problem of the minimum of the functional $I(u)$ on a set $M$ can be simplified if we know a functional $J(p)$, defined on some set $N$ and such that

$$\sup_{p \in N} J(p) = \inf_{u \in M} I(u).$$

The functional $J(p)$ is enlisted, for example, to construct bounds for the error in the approximate solution.

Particularly effective is the use of the functional $J(p)$ when we seek not the minimizing element but the minimal value $I_0$ of the functional $I(u)$. Computation of the elements of $I(u)$ and $J(p)$ at any element $u \in M$ and $p \in N$ gives upper and lower bounds for $I_0$

$$J(p) \leq I_0 \leq I(u).$$

If, for some $u$, we can choose $p$ such that $I(u)$ and $J(p)$ are close together, then $I_0 \approx I(u)$ and the error does not exceed $I(u) - J(p)$.

In this paper we indicate a method of constructing the functional $J(p)$ for a functional $I(u)$ of the form

$$I = E(u) - I(u^0),$$

$$E(u) = \int V U(z^i, u^i, \partial u^i/\partial x^i)dx,$$

$$I(u^0) = \int S F_u u^i dS;$$

where $V$ is a region in the n-dimensional space of the variables $x^i, u^i(x^i)$ are differentiable functions of the $x^i$, Latin superscripts run through the values $1, \ldots, n$, Greek superscripts run through the values $1, \ldots, m$, the function $U$ is convex and differentiable with respect to the variables $u^i$ and $\partial u^i/\partial x^i$, the $F_u$ are given functions of $x^i$ in $V$, the hypersurface $S$ is a part of the boundary $\partial V$ of the region $V$, the $F_u$ are given functions on $S$, and $dS$ is an element of the surface of $S$.

We seek a minimum of the functional $I$ on the functions $u^i$ which take given values on the surface $\Sigma = \partial V - S$:

$$u^i = q^i \text{ on } \Sigma.$$ (4)

When $E$ is a Dirichlet functional or a functional in the geometrically linear theory of elasticity, the variational principle formulated below goes over into the Thompson and the Castigliano principle respectively.

Consider the space $H_u$ of functions $\{u^i(x^i), u^{i\alpha}(x^k)\}$ (the $u^{i\alpha}$ are $m \times n$ independent functions of the $x^k$) and the functional $E$, defined in $H_u$:

$$E = \int V U(x^i, u^i, u^{i\alpha})dx.$$ (5)

On the set $L \subset H_u$, consisting of elements of the form

$$\{u^\alpha, \partial u^\alpha/\partial x^i\},$$ the functional $E$ coincides with the functional $E(u^\alpha)$. (2)

We introduce the $H_p$ of functions $\{p_\alpha(x^k), p^{i\alpha}(x^k)\}$ and let $\langle p \cdot u \rangle$ denote the bilinear form

$$\langle p \cdot u \rangle = \int V (p_u u^i + p^{i\alpha} u^{i\alpha})dx.$$ (3)

In $H_p$ the following functional is defined:

$$E'(p) = \sup_{u^0} (\langle p \cdot u \rangle - E),$$ (4)

which is Young's transformation of the functional $E$.

In $H_p$ we also construct the linear functional $l'(p)$ defined by the equation

$$l'(p) = \langle p \cdot u^0 \rangle - I(u^0),$$

where $\langle p \cdot u^0 \rangle$ is the value of the bilinear form $\langle p \cdot u \rangle$ on the set $L$,

$$\langle p \cdot u^0 \rangle = \int V (p_u u^i + p^{i\alpha} u^{i\alpha}/\partial x^i)dx,$$

and the $u^{i\alpha}$ are functions taking the values (4) on $\Sigma$.

Let $N$ denote the set in $H_p$ defined by the constraints

$$\langle p \cdot u^0 \rangle - I(u^0) = 0,$$ (5)

where $u^{i\alpha}$ are arbitrary functions taking zero values on $\Sigma$.

Obviously the values of the linear functional $l'(p)$ on $N$ are independent of the actual set of functions $u^{i\alpha}$.

Consider the problem of the maximum of the set $N$ of the functional

$$I(p) = l'(p) - E'(p).$$ (6)

We shall assume that there are constraints on the $U$, $F_u$, $f_\alpha$, $c_\alpha$ and the region $V$ such that solutions of the variational problems (1)-(4), and (6), (7) exist and are unique (cf. refs. 3-6).

To prove that the maximum value $I_0$ of the functional $I$ coincides with the minimum value $I_0$ of the functional $I$,

$$I_0 = J_0,$$ (8)

we require the following assertions:

1. Let $u_0$ be a fixed element of $H_u$ and $p_0$ be the maximizing element of the functional

$$\langle p \cdot u_0 \rangle - E'(p).$$ (9)

Then the maximizing element of the functional

$$\langle p \cdot u \rangle - E$$ (10)

in the space $H_u$ is $u_0$. 

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2. Put $u_0 = \{u_0^\alpha, \partial u_0^\alpha / \partial x^1\}$, where $u_0^\alpha$ is the minimizing element of the functional $I$ (1). Then the corresponding element $p_0$ satisfies the constraints (6).

Indeed, differentiating (10) in the direction at the point $u_0$ and putting $u = \{u^\alpha, \partial u^\alpha / \partial x^1\}$, we obtain

$$\langle p_0, \langle u^\alpha \rangle - DE(u_0^\alpha, u^\alpha) = 0,$$

where $DE(u_0^\alpha, u^\alpha)$ is the derivative of the functional $E(u^\alpha)$ at the point $u_0^\alpha$ in the direction $u^\alpha$. Using Euler's equation for the functional (1),

$$DE(u_0^\alpha, u^\alpha) - I(u^\alpha) = 0,$$

we obtain $\langle p_0, u^\alpha \rangle - I(u^\alpha) = 0$.

3. We have $E(u_0^\alpha) + E^*(p_0) = \langle p_0, u_0^\alpha \rangle$.

4. Differentiating the functional (9) at the point $p_0$ in the direction $p'$, we obtain

$$\langle p', u^\alpha \rangle - DE^*(p_0, p') = 0.$$  \hspace{1cm} (11)

5. The element $p_0$ is a maximizing element of the functional $J$.

Indeed, by 2, $p_0 \in N$. Further, let $p$ be an arbitrary element of $N$. Then $p$ can be put in the form $p = p_0 + p'$, where

$$\langle p', u^\alpha \rangle = 0.$$  \hspace{1cm} (12)

We can show that $J(p) \leq J(p_0)$.

$$J(p) = J(p_0) + DE^*(p_0, p').$$

Using (11) and (12), we obtain

$$J(p) = J(p_0) + \langle p', u^\alpha \rangle - \langle p', u^\alpha \rangle = J(p_0).$$

Consequently, $p_0$ is a maximizing element of the functional $J(p)$.

Equation (8) follows from 1-5:

$$I_0 = J(p_0) - E(p_0) = \langle p_0, u^\alpha \rangle - I(u^\alpha) - \langle p_0, u^\alpha \rangle + E(u_0^\alpha) = I_0 + \langle p_0, u^\alpha \rangle - I(u^\alpha) = I_0.$$

Note 1. We define the functional

$$L(p, u) = \langle p, u^\alpha \rangle - E^*(p) - I(u^\alpha).$$

The following equations are obvious:

$$\inf_{p_0} \sup_{u_0} L(p_0, u_0) = \inf_{p} \sup_{u} J(p, u) = I_0,$$

$$\sup_{p_0} \inf_{u_0} L(p_0, u_0) = \sup_{p} J(p, u) = I_0.$$  \hspace{1cm} (13)

where inf is taken over all the $u^\alpha$ satisfying the conditions (4).

Thus, the minimax problem (13) is equivalent to the problem of the minimum of the functional $I$ and its dual, the minimax problem (14), is equivalent to the problem of the maximum of the functional $J$.

Note 2. If the functions $p_0^\alpha$ are continuous and differentiable in the closed region $V$, the constraints (6) can be written as

$$\partial p_0^\alpha / \partial x^1 - p_0 + F = 0, \quad p_0^\alpha |_{S} = f_0;$$  \hspace{1cm} (15)

where the $n_1$ are the components of the external normal vector to $S$.

Note 3. If the function $U$ is independent of the $u^\alpha$, we have $E^*(p) = + \infty$ for $p_0 = 0$. Hence in seeking the maximum of $J$ we have to put $p_0 = 0$. The functional $E^*(p)$, for $p_0 = 0$, is the Young's transformation of the functional $E^*$ with respect to the variables $u^\alpha$.

Example 1. Thompson's principle. Suppose $V$ is the outside of a bounded region $\Omega$ in three-dimensional space and $E(u)$ is the Dirichlet functional,

$$E(u) = \int \frac{\partial u}{\partial x'} \frac{\partial u}{\partial x'} \, dx.$$

Consider the problem of the minimum of the functional $I = E(u)$ under the condition

$$u = 1 \text{ on } \partial V = \partial \Omega, \quad u(x) - c_1 / r + c_2 / r^2 + \ldots, \quad r = \sqrt{x'}.$$

The quantity $(2 \pi)^{-1} I_0$ is the electrostatic capacity of $\Omega$.

We know that

$$I_0 = \sup_{p} \left( \int_{\partial V} p \cdot n \, da \right) / 2 \int_{\partial V} p \cdot p \, dx,$$

where sup is taken over the whole vector field $p$ satisfying the equation

$$\partial p / \partial x^1 = 0.$$  \hspace{1cm} (18)

The variational problem (17), (18) is called Thompson's principle.

We can show that Thompson's principle follows from (7) and (8). In this case $S = 0, I(u) = 0, \Sigma = \partial V$. By Note 3, it is sufficient to consider certain fields $p$ satisfying (18), where

$$E^*(p) = \int \frac{\partial p}{\partial x'} \frac{\partial p}{\partial x'} \, dx.$$

Choosing a function $u$ which takes the boundary values (16), we obtain

$$I_0 = \sup_{p} \left( \int_{\partial V} p \cdot n \, da - \frac{1}{2} \int_{\partial V} p \cdot p \, dx \right).$$  \hspace{1cm} (19)

where sup is taken over vector fields $p$ satisfying (18).

We write $p$ as $p = \lambda p_0$, $\partial p_0 / \partial x^1 = 0$, where $\lambda$ is an arbitrary number, and rewrite (19) as follows:

$$I_0 = \sup_{p_0} \sup_{\lambda} \left( \int_{\partial V} p_0 \cdot n \, da - \lambda^{-1} \int_{\partial V} p_0 \cdot p_0 \, dx \right).$$  \hspace{1cm} (20)

After calculating sup, (20) takes the form of Thompson's principle.

Example 2. Castigliano's principle. Consider the functionals of the geometrically linear theory of
elasticity \( m = n = 3 \), and so in what follows we use only Latin letters:

\[
E = \int_{\Omega} U(x', \varepsilon_0) \, dx, \quad \varepsilon_0 = \varepsilon_{ij}^{\varepsilon} \frac{\partial a_i}{\partial x^j} + \frac{\partial a_j}{\partial x^i}.
\]

By Note 3, it is sufficient to seek the maximum of the functional \( J \) in the space of functions \( p^{ij} \), while \( E^* \) can be assumed to be Young's transformation of the functional \( E \) with respect to the variables \( u_{ij} \). However, since \( E \) depends only on the symmetric part \( \varepsilon_{ij} \) of the tensor \( u_{ij} \), \( E^* = + \infty \) for \( p^{ij} \neq p^{ij} \). Consequently, in seeking the maximum of \( J \) we have to put \( p^{ij} = p^{ij} \). The functional \( E^* \) then coincides with Young's transformation of the functional \( E \) with respect to the variables \( \varepsilon_{ij} \).

If the functions \( p^{ij} \) are continuous and differentiable in the closed region \( V \), the expression for \( J \) and the constraints (6) can be rewritten as

\[
J = \int_V p^{ij} n_{ij} \, dx - \int_{\gamma} U(x', p^{ij}) \, dx,
\]

(21)

\[
\partial p^{ij} / \partial x^j + F^j = 0 \quad \text{in} \ V, \quad p^{ij} n_{ij} = f \quad \text{on} \ S;
\]

(22)

where \( U^*(x^i, p^{ij}) \) is Young's transformation of the function \( U(x^i, \varepsilon_{ij}) \) with respect to the variables \( \varepsilon_{ij} \). The variational principle (21), (22) is known as Castigliano's principle.9

**Note 4.** The usual formulation of the principle (21), (22) is much more restricted than the formulation (6), (7) since it assumes that the functions \( p^{ij} \) are continuous and differentiable. Indeed, the maximum of \( J \) can be discussed over any summable functions \( p^{ij} \) (their values on sets of measure zero, in particular, on \( S \) and \( \Sigma \), are not defined).

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9O. A. Ladzhenskaya and N. N. Ural'tseva, Linear and Quasilinear Elliptic Equations [in Russian], Moscow (1964).


9E. G. Gol'dshtein, Duality Theory in Mathematical Programming and Its Applications [in Russian], Moscow (1971).
