Journal Title: Soviet Physics - Doklady
Volume: 22
Issue: 10
Month/Year: 1977
Pages: 604-606

Article Title: High frequency long wave vibrations of plates

Call #: Shelved as: Soviet physics, Doklady

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High-frequency long-wave vibrations of plates

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(Submitted June 21, 1977)


PACS numbers: 43.40.Dx

In the theory of plates and shells, the translation from the three-dimensional equations of the theory of elasticity to the approximate two-dimensional equations is possible if the displacements change little in the longitudinal directions over distances of the order of the thickness h. In statics, this condition leads, in the first approximation, to the constancy of displacements along the transverse coordinate. In dynamics, for stressed states which change slowly in the longitudinal directions, an infinite variety of types of displacement distribution along the thickness is possible. In the case of low frequencies ω(ρh/μ < 1, where ρ is the density and μ is the shear modulus) the displacements are, to a first approximation, constant along the thickness, just as in statics.

For high frequencies (ρh/μ ≥ 1) the displacements are rapidly oscillating functions of the transverse coordinate. The classical equations of the theory of plates and shells describe low-frequency vibrations. In this paper we derive two-dimensional equations for the description of free, high-frequency vibrations of plates.

1. Let us consider an isotropic, homogeneous plate of constant thickness h, referred to a Cartesian coordinate system x, xα (Greek indices take on the values 1 and 2). Let us position the average surface of the plate in the plane x = 0 and designate it by Ω. Let us designate the boundary of Ω by Γ, and the projections of the vector of displacements on the x, xα axes by w, wα. We will assume that the edge of the plate is rigidly fixed:

\[ w = w^α = 0 \text{ on } Γx. \]  

For free vibrations of an elastic plate, the variational equation of the mechanics of continuous media is holonomic and equivalent to the extremum principle for the functional

\[ I = \int dt \int Λ dx^4, Λ = U - \frac{1}{\rho} (w^α_t + w_x^α w^α_t), \]

\[ 2U = \lambda (w^α_x)^2 + 2μw^α_s \cdot w^α_s + (\lambda + 2μ) w^α_t^2 + μ(μ w^α_s + w^α_s) \cdot (w^α_t + w^α). \]  

Here, a comma in the indices denotes differentiation, and round brackets denote symmetrization.

2. We will obtain the equations for high-frequency vibrations by a variational–asymptotic method. Without loss of generality, we can assume that the displacements are bounded as h → 0. Let us make the substitution ξ = 2x/h and retain the terms in the functional (2) which dominate for small h:

\[ I = h \int \int Λ dx^4 dx^2 dt, Λ = \left( \frac{λ + 2μ}{h^2} w^α_t - ρ w^α_t + μ w^α_t + \frac{λ}{h^2} w^α_t \right) \]

\[ \text{We denote by } \langle \cdot \rangle \text{ the integral over } ξ \text{ with the limits } [-1, 1]. \]

The extrema of the functional I in Eq. (3) coincide with the extrema of the functional Λ. Varying Λ, we arrive at equations that describe four series of vibrations:

\[ F_1: \quad w = u \cos \alpha, w_x = 0, α = nπ, \]

\[ F_2: \quad w = 0, w_x = \psi \sin β, β = 1/2(2n + 1), \]

\[ L_1: \quad w = \psi \sin \alpha, w_x = 0, α = 1/2(2n + 1), \]

\[ L_2: \quad w = 0, w_x = \cos \beta, β = nπ. \]

Here \( α = \omega h/2c_1, β = ω h/2c_2, c_1 = \sqrt{(λ + 2μ)/ρ}, c_2 = \sqrt{μ/ρ}, α = \varphi, e = c_1/c_2, n \) is an integer, the quantities u, \( ψ_1, ψ_2, \psi_3 \) and \( w_0, w_1, w_2 \) are functions of \( x^α \) and \( n \) that are harmonic in time with frequency \( ω \); ω is determined by the values of \( α \) or \( β \) in the corresponding series. The sign \( \perp \) is used if the transverse displacement w is much larger than the longitudinal displacement \( w_0 \), and the sign \( \parallel \) is used in the opposite case. In each series there is an infinite number of types of oscillations (branches), corresponding to various values of \( n \). The branch of the series \( F_1 \) corresponding to a certain value of \( n \) is designated \( F_2(n) \). The designations of the branches of the remaining series are similar. We will assume that the functions u, \( u_0, ψ_1, ψ_2 \) satisfy the conditions

\[ w = \psi_3 = \psi_4 = 0 \text{ on } Γ. \]

3. Let us first consider the series \( F_1 \). Assuming u to be a given function of \( x^α \), we find \( w_0 \) [in the first approximation, according to Eq. (4), \( w_0 = 0 \)]. If we retain in Eq. (2) only the dominant terms containing \( w_0 \), and the dominant cross term, we arrive at the functional I in Eq. (3), in which Λ is given by the equation

\[ w = w_0 + \psi_0 = u_0 + \psi_0 = 0 \text{ on } Γ. \]
To derive Eq. (6) we performed an integration by parts and used the boundary conditions, Eq. (5). The extremum of the functional (6) is of the form

$$w = u + \frac{h}{2\mu} \left( \sin \alpha - \frac{2(1-e) \cos \beta}{\cos \beta} \right). \quad (2a)$$

Thus, $$w$$ is of the order of $$h$$. We similarly find the following correction to $$w$$: We fix $$w$$ and look for $$w$$ in the form $$w = u \cos \alpha + w$$, and retain in the Lagrangian the dominant terms in $$w$$ and the dominant cross term. In view of the overdetermination of $$u$$, we can impose on $$w$$ the constraint ($$w \cos \alpha$$) = 0. As a result, we find

$$w = \cos \alpha \left( \frac{h}{2\mu} \sin \alpha + \frac{2(1-e) \cos \beta}{\cos \beta} \right), \quad (5a)$$

4. In the same way, we obtain equations for the displacements of the remaining series:

$$F_i: \quad w = \psi \sin \frac{h}{2\mu} \left( \cos \alpha + \frac{2(1-e) \cos \beta}{\cos \beta} \right), \quad (5b)$$

$$L_i: \quad w = \psi \sin \alpha \left( \frac{h}{2\mu} \frac{\cos \beta}{\cos \beta} + \frac{2(1-e) \cos \beta}{\cos \beta} \right), \quad (5c)$$

5. If $$e$$ and $$n$$ are such that $$\cos \beta = \cos^{-1} n$$, then for the branch $$F_i$$ we must make the substitution $$u = h^{-1} \cos \beta$$ in the equations of paragraph 3. Then $$w = h \Delta \psi \psi \alpha \beta$$, $$w = m \alpha - e - e \sin \beta$$, where $$f(\alpha) = \cos \beta \psi (\alpha)$$ is a bounded function. We must make a similar substitution if $$\sin \alpha = 0$$ for any branch of $$F_i$$, $$\sin \beta = 0$$ for $$L_i$$, or $$\cos \alpha = 0$$ for $$L_k$$. In order to consider these cases together, we will assume $$e$$ to be irrational and if necessary carry out a limiting transition to rational values of $$e$$ in the final equations.

6. Let us now assume that in the equations of para-
\( \Gamma \) is free, then there is no orthogonality with respect to energy, and all forms of vibrations are present simultaneously.

8. In Eq. (9) there appears a more general property of orthogonality with respect to energy of the characteristic modes of vibration of the infinite plate.

Let us consider the characteristic modes of vibration of the form \( w = u(x) \exp i(k_\alpha x - \omega t) \), \( w = u(\zeta) \exp i(k_\alpha z - \omega t) \).

Substitution in the equations of motion gives

\[
\begin{align*}
    i(\mu + p^* u) + \omega^2 u &= 0, \\
    i(\mu + p^* u) + \omega^2 u &= 0; \\
    p^* = \zeta (k u + i f u) + \mu (k u + i f u), \\
    p^* = \mu (u + i k u); \\
    p = \zeta (k u + i f u) + 2 \mu; \\
    p = \mu (u + i k u) \text{ for } x = \pm h/2.
\end{align*}
\]

We will show that two solutions \( u, u^\alpha \) and \( u, (\mu + p^* (u, + i k u)) \), corresponding to one and the same real wave number \( k_\alpha \) and having different \( \omega \) and \( \omega' \) are orthogonal with respect to the kinetic and elastic energies. Let us multiply relations (10) by \( u^* \) and \( u^{\alpha*} \) (a bar denotes complex conjugation), integrate, and add. We obtain

\[
\omega^2 (u^{\alpha*} + uu^*) = (\mu + p^* u^* + p^* (u^{\alpha*} + i k u^{\alpha*} + p u^{\alpha*})) = 0.
\]

In particular, assuming \( u u^* = u^* u \) and \( u^* = u \), we find that the elastic and kinetic energies averaged over the thickness coincide, and that real values of \( k_\alpha \) correspond to real \( \omega \).

Let us interchange the primed and unprimed quantities in Eq. (11). Substitution of the equation so obtained from the complex conjugate of Eq. (11) leads to the relation \( (\omega^2 - \omega^2) (u_{\alpha}^{\alpha*} + uu^*) = 0 \), from which our assertion follows. It is significant that for other modes of vibration (for example, for complex \( k_\alpha \) or for real but different \( k_\alpha \)) there is generally no orthogonality with respect to energy.

Similar assertions are also valid in the problem of free vibrations of an elastic rod.

We note that the orthogonality of the branches with respect to energy permits a complete solution of the Cauchy problem for an infinite plate, strip, and rod, and also application of the energy methods developed by L. A. Kunin.

9. According to paragraph 6, free high-frequency vibrations of plates of the series \( F_L \) and \( F_\parallel \) are described by the equations

\[
\begin{align*}
    F_L: & \quad (\lambda + 2 \mu) (2h/k) \psi_x = \psi_x, \\
    F_\parallel: & \quad (\lambda + 2 \mu) (k + \mu) \psi_x + \psi_x = - (2 \psi/k) \psi_x - \lambda \psi_x - \mu \psi_x - \psi_x,
\end{align*}
\]

The equations for the series \( L_L \) and \( L_\parallel \) are obtained by substituting \( \psi = u \) and \( u^{\alpha} \) for \( \psi \) in Eq. (12).

Translated by P. J. Moxhay

Methods of extending the class of operators of the theory of viscoelasticity

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(Submitted by Academician Yu. N. Rabotnov, June 29, 1977)

PACS numbers: 03.40.0z, 46.30.0v

In this paper we will explain two ways (the \( T_H^* \) and \( T_H^{**} \) theories) of extending the class of hereditary operators of the classical theory \( T_H \).

1. In the \( T_H^* \) theory we will take as definitions any of the following relations:

\[
\begin{align*}
    \epsilon_{ij}(x, t) &= \int K_{ij} (x, t, x_0 (t)) d\omega, \\
    x_0 (t) &= k_o (x) = k_o (x) \delta x, \\
    F_{ij}(x, t) &= \int K_{ij} (x, t, x_0 (t)) d\omega, \quad (2) \quad (3) \quad (4)
\end{align*}
\]

Translated by P. J. Moxhay