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High-frequency vibrations of shells

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The equations of the theory of elastic shells can be regarded as equations obtained from the equations of the three-dimensional theory of elasticity in the long-wave approximation $h/\ell \to 0$, where $h$ is the thickness of the shell and $\ell$ is a characteristic spatial scale of the variation of the stressed state of the median surface (the *wavelength*). By their structure the equations of shell theory have physical significance only for $h/\ell \ll 1$. On the other hand, the formulation of boundary-value problems is associated with the behavior of the corresponding differential operator at short wavelengths $(h/\ell \to \infty)$, so that the formulation of the theory of shells involves not only the derivation of equations in the long-wave range, but also another logically independent step – the extrapolation of those equations to short waves.¹

It is possible to carry out either trivial extrapolations, when the system of equations derived for long waves is considered for short waves without any changes, or non-trivial extrapolations, when terms that are small in the long-wave range but appreciable for short waves are introduced into the system of equations. We clarify this by an example.

Let us consider the equations

$$\begin{align*}
\alpha^{2}_{h} u - c^{2}_{h} \alpha^{2}_{u} u &= 0, \\
\alpha^{2}_{\ell} u - c^{2}_{\ell} \alpha^{2}_{u} u + c^{2}_{h} \alpha^{2}_{u} u &= 0.
\end{align*}$$

(1)

Here $c$ is a constant having the units of velocity. In the long-wave range these equations are indistinguishable in the first approximation; the term $c^{2}_{h} \alpha^{2}_{u} u$ for long waves is small in comparison with the term $c^{2}_{\ell} \alpha^{2}_{u} u$ for short waves. In the short-wave range, however, Eqs. (1) differ fundamentally. The first equation is a wave equation requiring the formulation of two boundary conditions. In the second equation, for short waves the term $c^{2}_{h} \alpha^{2}_{u} u$, which is small in comparison with $c^{2}_{\ell} \alpha^{2}_{u} u$, can be rejected, so that the second equation is formed in the same way as the equation for the transverse vibrations of a beam and requires the formulation of four boundary conditions. Equations (1) can be regarded as two different short-wave extrapolations describing the same physical situation in the long-wave range.

For shells in the short-wave range it is impossible to describe the three-dimensional stressed state exactly by the two-dimensional theory, and only qualitative correspondence can be expected. For this reason, different two-dimensional equations are allowed in the theory of shells. It is natural to demand of the different short-wave extrapolations, however, asymptotic equivalence in the long-wave range.

In this article we propose a version of extrapolation to short waves of the equations formulated in Ref. 2 for the high-frequency vibrations of shells.

There are known²⁻⁶ to be a countable number of different vibrational modes (branches), and the corresponding distributions of the displacements with respect to the transverse coordinate $\xi (\xi \ll 1)$ for long waves in the first approximation have the form²⁺⁸

$$F_{1}(n): w = u \cos \alpha_{1}, \quad w_{0} = u_{\alpha_{0}} \frac{h}{2a} \left( \sin \alpha_{1} - \frac{2(-1)^{n} \sin(q\ell)}{\cos(q\ell)} \right), \quad \alpha = n\ell,$$

$$F_{1}(n): w = \psi_{1} \sin \beta_{1}, \quad w_{0} = \psi_{\beta_{0}} \frac{h}{2\beta} \left( \cos \beta_{1} - \frac{2(-1)^{n} \cos(q\ell)}{\sin(q\ell)} \right), \quad \beta = n\ell,$$

$$L_{1}(n): w = u \cos \alpha_{1}, \quad w_{0} = u_{\alpha_{0}} \frac{h}{2a} \left( -\sin \beta_{1} + \frac{2(-1)^{n} \sin(q\ell)}{\cos(q\ell)} \right), \quad \beta = n\ell,$$

$$L_{1}(n): w = \psi_{1} \sin \beta_{1}, \quad w_{0} = \psi_{\beta_{0}} \frac{h}{2\beta} \left( -\cos \alpha_{1} + \frac{2(-1)^{n} \cos(q\ell)}{\sin(q\ell)} \right). \quad \alpha = n\ell,$$

(2)

the symbols for the branches are written on the left, $w = w_{1} \sin \alpha_{1}, \quad w_{\alpha_{0}} = w_{1} \sin \beta_{1}$, $w_{\alpha_{0}} = w_{1} \sin \beta_{1}$ (the projections of the displacements $w_{1}$ onto the normal to the tangent vectors $\ell_{1}, \ell_{2}, \ell_{3}$, $\alpha_{0}, \beta_{0}$, $\gamma_{0}$ are certain functions of the coordinates $\xi$ on the median surface $\Omega$ and of the time $t$ (distinct for each branch; the branch-numerating index is omitted here so as not to encumber the notation too much); a comma before a Greek subscript denotes differential with respect to $\xi$, and a semicolon denotes covariant differentiation along $\Omega$; $e = \sqrt{\mu/\ell(\lambda + 2\mu)}$; $\lambda, \mu$ are the Lamé coefficients. Correction terms of order $h/\ell$ R in comparison with unity are not written in (2) (they are given in Ref. 2; $R$ is the characteristic radius of curvature of $\Omega$). The distributions of the displacements with respect to $\xi$ for the classical branches $F_{1}(0)$ and $L_{1}(0)$ are the same as in statics.”¹⁰⁻¹¹

It has also been shown²⁻⁹ that in the long-wave range under the condition of a rigidly built-in edge of the shell the action functional decomposes into a sum of functionals, each of which depends on the field functions of only one branch, so that the equations for all the branches (except the classical branches describing low-frequency vibrations) are independent.¹¹

The equations derived in Ref. 2 are asymptotically exact and correctly describe the behavior of shells in the long-wave range. However, the trivial extrapolation carried out in Ref. 2 yields an unsatisfactory description of the dispersion curves and the group velocities in the short-wave range. The reason for this lies in the fact that trivial extrapolation makes the branches orthogonal with respect to the energy in the short-wave range and excludes interaction between them; this is too coarse
an approximation. Below, we give a different short-wave extrapolation, which takes account of the interaction of branches with \( n = 0, 1 \).

We consider vibrations that can be regarded with sufficient accuracy as the superposition of vibrations of the lowest-frequency branches \( F_{1}(0), F_{2}(0), L_{1}(0), L_{2}(1) \). The branch \( F_{1}(0) \) corresponds to low-frequency bending vibrations, \( F_{2}(0) \) to high-frequency bending vibrations with the lowest frequency, \( L_{2}(0) \) to low-frequency longitudinal vibrations, and \( L_{1}(0) \) and \( L_{2}(1) \) to the two high-frequency longitudinal modes with the lowest frequencies. The need to introduce the two branches \( L_{1}(0) \) and \( L_{2}(1) \) simultaneously is associated with the fact that they interact strongly, as is evident from the pattern of the dispersion curves for longitudinal vibrations of a plate (Fig. 1).

The dynamic equations contain eight unknown functions of the coordinates and the time: \( \bar{u}(x, t), \bar{v}(x, t), \bar{w}(x, t), \bar{n}(x, t), \bar{a}(x, t), \bar{\alpha}(x, t), \bar{\beta}(x, t), \bar{\gamma}(x, t) \), describing the branches \( F_{1}(0), F_{2}(0), L_{1}(0), L_{2}(1), L_{2}(0) \) (the corresponding symbols without the bar are reserved for functions that arise after certain substitutions for the unknown functions in the final equations). Despite the fact that the theory involves more unknown functions than in the classical theory of shells, it has the meaning, not of a refined theory, but of a first-approximation theory, because it describes asymptotically exactly the vibrations of a shell in the range of low waves and high frequencies (\( \omega \geq 2\pi c_{2}/\beta, c_{2} = \sqrt{\mu/\beta} \)).

Thus, we assume that the displacements have the form

\[ w_{a} = \bar{u}_{a} - n_{i}(x)u \frac{h}{2} + \bar{v}_{a} \sin \frac{\pi x}{2} + \bar{v}_{a} \cos \frac{\pi x}{2} + \bar{w}_{a} \frac{h}{2} \left( \cos \frac{\pi x}{2} + \sin \frac{\pi x}{2} \right) + 2 \cos \frac{\pi x}{2} \left( \frac{\sin \frac{\pi x}{2}}{\cos \frac{\pi x}{2}} \right), \]

\[ w = \bar{u} - a_{\alpha} \frac{h}{2} - \bar{v}_{\alpha} \frac{h}{2} \left( t^{2} - \frac{1}{3} \right) + \frac{h}{8} \sin \frac{\pi x}{2} - \bar{v}_{\alpha} \frac{h}{2} \left( \sin \frac{\pi x}{2} \right), \]

where

\[ \lambda_{a} = \rho_{a} \left( \bar{u}_{a} + \bar{v}_{a} \right) \]

are measures of the extension and bending, \( \sigma = \sqrt{\lambda (\lambda + 2 \mu)} \), and \( \beta_{a} \) are the components of the second quadratic form of \( \Omega \).

We substitute the formulas (3) into the action functional of an elastic shell:
Varying with respect to $u_\alpha$, $\nu$, $\psi$, we obtain the equations

$$
-n^\alpha_\beta - \left( m^\alpha_\beta \tilde{u}^\beta \right)_\beta + \frac{\mu}{2} \left( \frac{\pi^2}{24} \right) \left( \psi_{,\alpha} + u_{,\beta} + u_{,\alpha} + \kappa_\alpha \right) + \rho \tilde{u}^\alpha_{,tt} = \frac{1}{h} \left[ \tilde{p}^\alpha \right],
$$

$$
-n^\alpha_\beta b_{\alpha\beta} - \frac{\mu}{2} \left( \frac{\pi^2}{24} \right) \left( \psi_{,\alpha} + u_{,\alpha} + \kappa_\alpha \right) + \rho \tilde{u}^\alpha_{,tt} = \frac{1}{h} \left[ \tilde{p}^\alpha \right].
$$

$$
-m^\alpha_\beta \tilde{u}^\beta + \left( \frac{\mu}{2} \frac{\pi^2}{24} \right) \left( \psi_{,\alpha} + u_{,\alpha} + \kappa_\alpha \right) + \frac{\rho}{2} h^2 \left( \frac{\pi^2}{24} \right) \tilde{u}_{,tt} = \frac{1}{h} \left[ \tilde{p}^\alpha \right].
$$

$$
n^\alpha_\beta = 2\mu (\sigma + \sigma^\alpha + \gamma), \quad m^\alpha_\beta = \mu \frac{h^2}{6} (\sigma + \sigma^\alpha + \gamma).
$$

(7)

For plates ($b_{\alpha\beta} = 0$) the functional (5) decomposes into a sum of two functionals, of which one depends on $u$ and $\psi$ and corresponds to transverse vibrations, while the other depends on $u^\alpha$, $\psi$, and $v^\alpha$ and corresponds to longitudinal vibrations. We give the Euler equations for these functionals:

$$
-\frac{\mu}{2} \left( \frac{\pi^2}{24} \right) \left( \psi_{,\alpha} + u_{,\alpha} \right) + \rho \tilde{u}^\alpha_{,tt} = \frac{1}{h} \left[ \tilde{p}^\alpha \right].
$$

$$
-\mu (2\alpha + 1) \frac{h^2}{12} \left( \psi_{,\alpha} + u_{,\alpha} \right) + \frac{\mu}{2} \left( \frac{\pi^2}{24} \right) \left( \psi_{,\alpha} + u_{,\alpha} \right) + \frac{\rho}{2} h^2 \left( \frac{\pi^2}{24} \right) \tilde{u}_{,tt} = \frac{1}{h} \left[ \tilde{p}^\alpha \right].
$$

$$
-2\mu (2\alpha + 1) \psi_{,\alpha \alpha} - 2\mu u_{,\alpha} - \mu \frac{\pi^2}{h^2 e^2} \left( \psi + 4h \tilde{u}_{,\alpha} \right) \tilde{u}_{,\alpha} + 2\rho \tilde{u}_{,tt} = \frac{1}{h} \left[ \tilde{p}^\alpha \right].
$$

(8)

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(8)

Translating by J. S. Wood