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ON THE THEORY OF CURVILINEAR TIMOSHENKO-TYPE RODS*

V.L. BERDICEVSKII and L.A. STAROSEL'SKII

An asymptotically exact theory of isotropic Timoshenko-type rods is developed. The variational problem is formulated for the cross-section in order to calculate the transverse shear coefficients. Shear coefficients are obtained for a series of transverse cross-sections.

1. One-dimensional theory of rods. In classical theory a rod is modelled by a curve \( \Gamma \) with a set of orthogonal reference vectors attached at each point, with one vector of each set tangential to \( \Gamma \). The theory allows for four functionally independent degrees of freedom: three components \( r^i(\xi) \) of the radius vector of the points on \( \Gamma \) (the small latin letters \( i,j,k \) take the values 1,2,3 and correspond to the projections on the Cartesian axes of the observer coordinate system, and \( \xi \) is a parameter on \( \Gamma \)) and six components of the reference vectors \( \tau_{ab}^i \) orthogonal to \( \Gamma \) (the small greek letters take the values 1,2) connected by the following five relations:

\[
\tau_{ab}^i \tau_{ab} = \delta_{ab}, \quad \tau_{ab}^i r_{ab}^i = 0 \tag{1.1}
\]

where \( \delta_{ab} \) are the Kronecker deltas and the comma preceding \( \xi \) in the subscripts denotes differentiation with respect to \( \xi \). The degree of freedom which exists when the set of reference vectors is defined, describes the relative rotation of the transverse cross-section. The curvatures \( \omega_a \) and torsion \( \omega_b \) of the rod are given by the relations

\[
\begin{align*}
\gamma &= \frac{1}{2} (r^i_{,i} - 1), \quad \Omega_a = (1 + 2\gamma) r^i_{,i} \omega_a - \omega^a, \\
\Omega &= (1 + 2\gamma) \omega - \omega^0
\end{align*}
\]

The superscript \( ^0 \) denotes quantities in the undeformed state, and we use the arc length on the curve \( \Gamma_0 ^0 \) as the parameter \( \xi \). The formulas for \( \gamma, \Omega_a \) and \( \Omega \) are written in terms of \( \tau_{ab}^i (\xi) \) and \( r^i(\xi) \) as follows:

\[
\begin{align*}
\gamma &= \frac{1}{2} (r^i_{,i} r_{ab}^i - 1), \quad \Omega_a = r^i_{,i} \tau_{ab}^i - \omega_a, \quad \Omega = \frac{1}{2} \epsilon_{abc} \tau_{ab}^i - \omega^0 \tag{1.2}
\end{align*}
\]

The variational equation of the one-dimensional theory of rods has the form

\[
\frac{\delta}{\delta \xi} \left( \int_0^{\xi} K \, d\xi \right) + \frac{\delta}{\delta \xi} \left( \int_0^{\xi} \rho A \, d\xi \right) = 0 \tag{1.3}
\]

where \( |\Gamma_0^0| \) is the rod arc length in the undeformed state, \( K \) and \( \Phi \) are the kinetic and internal energy density per unit length, \( A \) is the work done by external forces, and \( r^i \) and \( \tau_{ab}^i \) are the functions varied and obeying the relations (1.1).

In the classical theory of isotropic inhomogeneous rods (with centrally symmetric cross-section and even elastic properties) we have

\[
\begin{align*}
2\Phi &= \langle E \rangle \gamma^2 + \langle E_{E_s} \rangle \Omega_a \Omega_a + C \Omega^2, \\
2K &= \langle \rho \rangle r^i r_{,i} + \langle E_{E_s} \rangle \tau_{ab}^i \tau_{ab}^i \tag{1.4}
\end{align*}
\]

Here \( \langle \cdot \rangle \) denotes the integral over the transverse cross-section, \( \xi^2 \) are the Cartesian coordinates in the transverse cross-section, \( E \) is Young's modulus and \( \rho \) is the density of the material. To compute the torsional rigidity \( C \) we must solve the Saint-Venant problem at the cross-section \( (\mu \) is the shear modulus and the comma preceding the greek subscripts denotes differentiation with respect to \( \xi^a \))

Here $α$ are the Levi-Civita symbols with $α = 0$, $α = -α = 1$.

Formulas (1.4) were obtained for a homogeneous rod from the Kirchhoff hypothesis on plane flows \( f/3 \). They can be obtained by an asymptotic analysis of a three-dimensional energy functional, retaining in the expression for the energy the principal terms only and neglecting corrections of the order of \( b/\varepsilon \) and \( b \) compared with unity \( \varepsilon/2 \), where \( b \) is the diameter of transverse cross-section, \( \varepsilon \) is the characteristic radius of torsional curvature, \( \varepsilon \) is the characteristic scale of change in the stress state along the rod and \( \varepsilon \) is the deformation amplitude (the quantities \( b, \varepsilon \) are defined in (2/2)). Below we construct a more accurate theory, in which the expression for the energy retains corrections of the order of \( b/\varepsilon \) and \( b/\varepsilon^2 \) compared with unity.

In the improved theory the vectors \( \mathbf{v}_\alpha \) are assumed to be orthogonal to the vector \( \mathbf{v}_\alpha \) and two additional degrees of freedom are introduced, namely the transverse shear \( \mathbf{v}_\alpha = \mathbf{v}_{\alpha \beta} \) (this idea is due to I.P. Yaroshenko [1]). The vectors \( \mathbf{v}_\alpha \) are of unit length and mutually orthogonal \( \mathbf{v}_\alpha \cdot \mathbf{v}_\beta = \varepsilon \delta_{\alpha \beta} \) just as in the classical theory. The measures of fluxure and torsion are found in terms of the vectors \( \mathbf{v}_\alpha \), \( \mathbf{v}_\alpha \) by means of the formulas (1.2). The internal energy density contains, in the improved theory, apart from the classical terms, the cross terms connecting the elongation with torsion, elongation with fluxure, torsion with shear energy:

\[
\mathcal{E}_\varepsilon = \varepsilon^2 \left( b^2 \mathbf{v}_\alpha \dot{\mathbf{v}}_\alpha + \mathbf{D}_{\mathbf{v}} \mathbf{D}_{\mathbf{v}}^\alpha + \mathbf{D}_{\mathbf{v}} \mathbf{D}_{\mathbf{v}}^\beta + \mathbf{C}_{\mathbf{v}} \right) + \varepsilon^3 \left( b^2 \mathbf{v}_\alpha \dot{\mathbf{v}}_\alpha - \mathbf{D}_{\mathbf{v}} \mathbf{D}_{\mathbf{v}}^\alpha \right) - 2 \varepsilon \left( b^2 \mathbf{v}_\alpha \dot{\mathbf{v}}_\alpha - \mathbf{D}_{\mathbf{v}} \mathbf{D}_{\mathbf{v}}^\alpha \right)
\]

The cross terms connecting the elongation with torsion exists only for naturally twisted rods \( \mathbf{v}_\alpha \neq \mathbf{0} \) and was first computed in [6/4], while in \( \varepsilon/5 \) it was obtained from the asymptotic representations. The quantity \( \mathcal{E}_\varepsilon \) characterizes the interaction between the elongation and torsion and is computed from the moduli of rigidity of the classical theory \( \mathcal{E}_\varepsilon \) and \( \mathcal{C} \). In contrast, the effect of transverse shear is connected with three conditional independent characteristics of the rod, namely with the shear rigidities \( \mathcal{J}_{\mathbf{v}} \). Below we show that the latter is obtained from the solution of the following variational problem at the cross-section:

\[
\mathcal{J}_{\mathbf{v}} = \min \left\{ \mathcal{J}_{\mathbf{v}} : (\mathbf{v}_\alpha \neq \mathbf{0}) \right\}
\]

The minimum in (1.7) is sought over all functions \( \mathbf{v}_\alpha \) satisfying the restriction \( \mathbf{v}_\alpha = \mathbf{0} \), \( \mathbf{v}_\alpha \) are orthogonal to \( \mathbf{v}_\alpha \) and the energy \( \mathcal{E}_\varepsilon \) represents formally the principal part of the energy. Therefore, in the first step of the variational-asymptotic method \( \varepsilon/6 \) it is necessary to find the energy in accordance with the hypothesis of plane flows \( \mathbf{v}_\alpha = \mathbf{0} \) in the first approximation. However, \( \mathbf{v}_\alpha \neq \mathbf{0} \) (one such case is discussed below) when \( \mathbf{v}_\alpha \neq \mathbf{0} \), and the theory derived below, regarded as improved, becomes a first approximation theory.

The rods whose curvilinear in the undeformed state are characterized by another five parameters, i.e., four components of the non-symmetric tensor \( \mathbf{D}_{\mathbf{v}} \) and scalar \( \mathcal{C} \), connected with the cross relation between the elongation and fluxure, and fluxure and torsion. To find \( \mathcal{J}_{\mathbf{v}} \) we must obtain the coefficients of the quadratic form \( \mathcal{J}_{\mathbf{v}} = \mathbf{v}_\alpha \dot{\mathbf{v}}_\alpha + \mathbf{D}_{\mathbf{v}} \mathbf{D}_{\mathbf{v}}^\alpha + \mathcal{C}_{\mathbf{v}} \mathbf{v}_\alpha \mathbf{v}_\alpha \), representing the minimum value of the functional:

\[
\mathcal{L}_{\mathbf{v}} = \frac{1}{2} \left\{ \mathbf{D}_{\mathbf{v}} \mathbf{D}_{\mathbf{v}}^\alpha + \mathbf{C}_{\mathbf{v}} \mathbf{v}_\alpha \mathbf{v}_\alpha \right\}
\]

where \( \mathbf{v}_\alpha = \mathbf{v}_\alpha \) and \( \mathbf{v}_\alpha \) are components of the metric tensor components. The components of the strains \( \mathbf{v}_\alpha \), \( \mathbf{v}_\alpha \) are determined from the condition of the moment of the energy

\[
\mathcal{L}_{\mathbf{v}} = \frac{1}{2} \left\{ \mathbf{D}_{\mathbf{v}} \mathbf{D}_{\mathbf{v}}^\alpha + \mathbf{C}_{\mathbf{v}} \mathbf{v}_\alpha \mathbf{v}_\alpha \right\}
\]

The problem in question reduces to replacing the three-dimensional problem of the theory of the rod's stability by the appropriate "one-dimensional" problem containing the functions of the elongation, shear energy and the action of the external forces and the condition of the one-dimensional theory may be regarded as a simplification of this problem. To construct the one-dimensional theory we use the variational-asymptotic method \( \varepsilon/7 \).

Asymptotic analysis of the three-dimensional problem. Transformation of the expression for the energy. We will write \( \mathcal{L}_{\mathbf{v}} \) in the form of a sum of three positive definite quadratic forms, i.e., the longitudinal energy \( \mathcal{L}_{\mathbf{v}} \), the transverse energy \( \mathcal{L}_{\mathbf{v}} \), and the shear energy \( \mathcal{L}_{\mathbf{v}} \).

\[
\mathcal{L}_{\mathbf{v}} = \mathcal{L}_{\varepsilon} + \mathcal{L}_{\varepsilon} + \mathcal{L}_{\varepsilon}
\]

Here the symbol \( \varepsilon = \varepsilon \) describes the expression within the brackets in which the subscript \( \alpha, \beta \) have been replaced by \( \gamma, \delta \). The components of the "two-dimensional" electromagnetic tensors are expressed in terms of the metric tensor components by means of the Poisson's ratio \( \nu \).
\[ G = 4 \beta (x^2 + y^2) + (2 \beta x^2 - y^2) \beta^2 \]
\[ E_3 = -\frac{1}{E_3} \Delta E_3 \]

Remembering that the metric tensor components are given in the $s$ coordinate system by the formulas:

\[ \Delta = \begin{pmatrix} 1 + \alpha \beta^2 \gamma & \alpha \beta^2 \gamma & 0 \\ \alpha \beta^2 \gamma & 1 + \alpha \beta^2 \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
\[ \Delta E_3 = -\frac{1}{E_3} \Delta E_3 \]

and neglecting terms of the order of $(h/R)^2$ compared with unity, we obtain the following expressions for the components of the two-dimensional elastic modulus:

\[ E_1 = E(1 - 4 \alpha \beta^2 \gamma) \]
\[ E_2 = E \Delta E_2 \]
\[ E_3 = E \Delta E_3 \]
\[ E_4 = E \Delta E_4 \]
\[ E_5 = E \Delta E_5 \]
\[ E_6 = E \Delta E_6 \]

External forces.

We will assume that the external surface and mass forces are of the order of

\[ P_i = \frac{1}{\lambda} \frac{h}{T} \]

and the mass forces $F_i$ are constant over the cross-section $S$.

Let us assume that the relation $P_i (y, z)$ at the end faces can be written in the form $P_i = (C_i + E \Delta C_i) y$ where $C_i, E \Delta C_i = const$. We begin the asymptotic analysis of the three-dimensional problem by considering the static case.

First approximation. As was shown in [10], the law of motion to a first approximation has the form

\[ x (r, \theta) = r \sin \theta \]
\[ y (r, \theta) = r \cos \theta \]
\[ z (r, \theta) = \frac{1}{2} \frac{h}{T} \sin \theta \]

where $y$ is the mixing element in the variational problem (1.5). Substituting (1.1) into the expression for the energy and integrating over the transverse cross-section, we obtain the formula for $q (r, \theta)$. Subsequent terms of the expansion (1.1) are of the order of $(h/R)^2$ and make only a small contribution to the energy.

Second approximation. In accordance with the general scheme of the variational-asymptotic method we write the law of motion in the form

\[ x (r, \theta) = r \sin \theta \]
\[ y (r, \theta) = r \cos \theta \]
\[ z (r, \theta) = \frac{1}{2} \frac{h}{T} \sin \theta \]

The arbitrary choice of $r_i$ and $y_i$ makes it possible to impose the following restrictions on $x_i$:

\[ \beta_i = 0, \beta_i \zeta_i = 0, \beta_i y_i = 0, \beta_i z_i = 0 \]

where $\beta_i$ denotes differentiation with respect to $y_i$. Let us substitute (2.1) into the expression for the strain tensor components (2.3). Neglecting quantities of the order of $(h/R)^2$ and $\alpha (h/R)^2$, we obtain

\[ \varepsilon = \frac{1}{(1 - \nu)4\pi} (28 \pi R^2 + 28 \pi R^2 \zeta_i + 28 \pi R^2 \zeta_i y_i + 28 \pi R^2 \zeta_i z_i) \]

Let us separate, from the energy functional, the terms containing $\zeta$ and $\zeta$, principal in the asymptotic sense. By (3.4) the terms have the form

\[ \frac{1}{(1 - \nu)4\pi} (28 \pi R^2 + 28 \pi R^2 \zeta_i + 28 \pi R^2 \zeta_i y_i + 28 \pi R^2 \zeta_i z_i) \]

Let us carry out the following substitution of the function sought:

\[ \zeta = \zeta + \frac{1}{(1 - \nu)4\pi} (28 \pi R^2 + 28 \pi R^2 \zeta_i + 28 \pi R^2 \zeta_i y_i + 28 \pi R^2 \zeta_i z_i) \]

In terms of the functions $\zeta, \zeta$, the principal terms take the form

\[ \frac{1}{(1 - \nu)4\pi} (28 \pi R^2 + 28 \pi R^2 \zeta + 28 \pi R^2 \zeta_i y + 28 \pi R^2 \zeta_i z_i) \]

The restrictions (3.1) for the function $\zeta'$ become

\[ \frac{1}{(1 - \nu)4\pi} (28 \pi R^2 + 28 \pi R^2 \zeta_i + 28 \pi R^2 \zeta_i y_i + 28 \pi R^2 \zeta_i z_i) = 0 \]

Thus at the second step of the variational-asymptotic method the law of motion is determined $y$ and to including terms of the order of $(h/R)^2$. It is shown that in this case it is sufficient to construct a theory including in the expression for the energy corrections of the order of $(h/R)^2$ and $(h/R)^3$ compared with unity.

Substituting (3.1) into the variational equation (3.2) and retaining terms of the necessary order, we obtain the variational equation (3.1) with the densities of internal energy and work done by the forces of the form

\[ \Phi = \frac{1}{(1 - \nu)4\pi} (28 \pi R^2 + 28 \pi R^2 \zeta_i + 28 \pi R^2 \zeta_i y_i + 28 \pi R^2 \zeta_i z_i) \]

In the homogeneous case the formula for $\Phi$ when $\gamma = 0$ was obtained in [1/1] and in the framework of the linear theory of rectilinear rods, in fact, in [2/1]. However, neither in [1/1] nor in [2/1] was the possibility mentioned of the transformation carried out below, and $\gamma$ (1.3) was not reduced to its final simple form (1.6).

Transformation of the variational equation. We will simplify the expression for the energy by carrying out a substitution of the functions sought. We will redefine the transformation axes

\[ \theta = \theta + \lambda \gamma \zeta_i \]

where $\lambda \gamma$ is a constant, selected, in what follows, in a special manner. Then the new measure can be rewritten thus
\[ \Omega_\nu = \Omega_\nu + 2\delta_0 \Omega_\nu + \Omega_\nu - \nu \Omega_\nu - \nu \Omega_\nu \] (3.10)

Taking into account (3.10) we can write the group of terms from the expression for the energy density \( \Phi \) in the form

\[ \frac{1}{2} (\text{C} \mu - \text{C} \mu)^T \mu \nu + \frac{1}{2} (\text{C} \mu - \text{C} \mu)^T \mu \nu + \frac{1}{2} (\text{C} \mu - \text{C} \mu)^T \mu \nu + \frac{1}{2} (\text{C} \mu - \text{C} \mu)^T \mu \nu \]
(3.11)

\[ \frac{1}{2} (\text{C} \mu - \text{C} \mu)^T \mu \nu + \frac{1}{2} (\text{C} \mu - \text{C} \mu)^T \mu \nu + \frac{1}{2} (\text{C} \mu - \text{C} \mu)^T \mu \nu + \frac{1}{2} (\text{C} \mu - \text{C} \mu)^T \mu \nu \]

where we omit the divergent term \( (\text{C} \mu - \text{C} \mu)^T \mu \nu \). Let us write the sum on the right-hand side of (3.11) as a quadratic form in \( \nu \), \( \nu + \text{C} \mu \)

\[ \nu + \text{C} \mu + \text{C} \mu \nu + \frac{1}{2} \nu \text{C} \mu + \frac{1}{2} \nu \text{C} \mu + \frac{1}{2} \nu \text{C} \mu + \frac{1}{2} \nu \text{C} \mu \nu + \frac{1}{2} \nu \text{C} \mu + \frac{1}{2} \nu \text{C} \mu + \frac{1}{2} \nu \text{C} \mu + \frac{1}{2} \nu \text{C} \mu \]

The tensors \( \text{C} \mu \), \( \text{C} \mu \) and \( \text{C} \mu \) are connected by the equations

\[ \nu + \text{C} \mu + \text{C} \mu \]
(3.12)

Solving the first relation of (3.12) for the tensor \( \text{C} \mu \) and substituting the result in the same relation, we obtain

\[ \nu + \text{C} \mu + \text{C} \mu \]

We satisfy the relation (3.13) by an appropriate choice of the tensor \( \lambda \), which we shall assume to be symmetric. Substituting into (3.13) the values of the terms appearing in it as given by (3.11), we obtain

\[ (\text{C} \mu - \text{C} \mu)^T \mu \nu = \frac{1}{2} (\nu - \frac{1}{2} (\text{C} \mu + \text{C} \mu)) \]
(3.14)

Expression (3.14) represents a system of three linear equations for the components of the tensor \( \lambda \). It can be shown that the determinant of (3.14) is not zero. Let us suppose that the axes of the associated coordinate system \( \nu \) coincide with the axes of the cross-section \( \Sigma \) in which the tensor \( (\text{C} \mu - \text{C} \mu)^T \mu \nu \) is diagonal. Then (3.14) separates into three independent equations which yield the values of \( \lambda \) (1.9). Let us carry out another substitution of the functions sought, namely \( \nu = \nu + \text{C} \mu \)

Replacing in the linear measure \( \Omega_\nu \) by \( \nu \), we obtain

\[ \Omega_\nu = \Omega_\nu + \nu \text{C} \mu + \Omega_\nu \]
(3.15)

The expression for the linear measure will be the same as the earlier expression if we redefine simultaneously the components of the radius vector \( r' \), \( r' = r' + \nu \text{C} \mu \). We note that within the accuracy used we can also represent the substitution of the unknown functions in the form

\[ r' = r' + \nu \text{C} \mu \]
(3.16)

We will now write the expression for the linear measure with an accuracy of the order of \( \text{C} \mu \) and \( \text{C} \mu \) as follows

\[ \Omega_\nu = \Omega_\nu + \nu \text{C} \mu + \nu \text{C} \mu + \nu \text{C} \mu \]

Let us write the group of terms in the energy density connected with the torsion, in the form

\[ \frac{1}{2} (\text{C} \mu - \text{C} \mu)^T \mu \nu + \frac{1}{2} (\text{C} \mu - \text{C} \mu)^T \mu \nu + \frac{1}{2} (\text{C} \mu - \text{C} \mu)^T \mu \nu + \frac{1}{2} (\text{C} \mu - \text{C} \mu)^T \mu \nu \]

where we omit the divergent term \( (\text{C} \mu - \text{C} \mu)^T \mu \nu \). Let us write the sum on the right-hand side of (3.11) as a quadratic form in \( \nu \), \( \nu + \text{C} \mu \)

\[ \nu + \text{C} \mu + \text{C} \mu \nu + \frac{1}{2} \nu \text{C} \mu + \frac{1}{2} \nu \text{C} \mu + \frac{1}{2} \nu \text{C} \mu + \frac{1}{2} \nu \text{C} \mu \]

(3.17)

From (3.17) it follows that

\[ \nu = \nu + \nu \text{C} \mu + \nu \text{C} \mu + \nu \text{C} \mu \]

(3.18)

Taking into account (3.16) and (3.18), we can write the expressions for the linear measure \( \Omega_\nu \) and elongation of the \( \gamma \)-axis in the form

\[ \Omega_\nu = \Omega_\nu + \nu \text{C} \mu + \nu \text{C} \mu + \nu \text{C} \mu \]

\[ \nu = \nu + \nu \text{C} \mu + \nu \text{C} \mu + \nu \text{C} \mu \]

After these transformations and substitutions of the unknown functions, the energy density \( \Phi_\nu \) is reduced (the bars are omitted) to the form (3.9) with different values of the effective forces \( K \) and \( \text{H} \)

\[ K = \frac{1}{2} (\text{C} \mu + \text{C} \mu)^T \mu \nu + \nu \text{C} \mu + \nu \text{C} \mu + \nu \text{C} \mu \]

\[ \text{H} = \frac{1}{2} (\text{C} \mu + \text{C} \mu)^T \mu \nu + \nu \text{C} \mu + \nu \text{C} \mu + \nu \text{C} \mu \]

The work done by the forces at the ends \( \mu, \nu, \lambda \) in the same in the improved theory as in the classical theory. In statics, the correctness of this step is guaranteed by the Saint-Venant principle. In dynamics, the problem of the boundary conditions requires special investigation.

4. Effective coefficients of the one-dimensional theory. Below we give the value of the coefficients \( \mu \) and \( \text{H} \) and the corresponding minimizing functions of the variational problem (1.6) in the one-dimensional case for certain transverse cross sections.

1o. A circle of radius \( r \)

\[ \mu = \frac{1}{2} \left( r \right)^2 \mu, \quad \text{H} = \frac{1}{2} \left( r \right)^2 \text{H} \]

2o. An annulus with radii \( r \) and \( r \) (\( r \) < \( r \))

\[ \mu = \frac{1}{2} \left( r \right)^2 \mu, \quad \text{H} = \frac{1}{2} \left( r \right)^2 \text{H} \]

3o. An ellipse \( (r \) < \( r \))

\[ \mu = \frac{1}{2} \left( r \right)^2 \mu, \quad \text{H} = \frac{1}{2} \left( r \right)^2 \text{H} \]

4o. A rectangle \( |x| < L \) and \( |y| < L \)

\[ \mu = \frac{1}{2} \left( r \right)^2 \mu, \quad \text{H} = \frac{1}{2} \left( r \right)^2 \text{H} \]
The quantities $J_2 = \frac{1}{2} M_2$ and the functions $q_1, q_2$ for the ellipse and rectangle are obtained by making the substitutions $s^2 = b^2$ and the change of indices $1 \rightarrow 2$. The remaining components of $\frac{p^2}{a^2}, \frac{p^2}{b^2}$ are zero with respect to the principal axes of inertia.

$5$. An inhomogeneous rod of rectangular transverse cross section $[s_1, s_2] \times [b_1, b_2]$, with shear modulus $\mu$ depending arbitrarily on the coordinate $z$,

$$f_1 = \psi q_1^2, \quad f_2 = \psi q_2^2, \quad s = \psi q_1^2, \quad \psi = \psi(x, y)$$

The integration constants are fixed by the conditions $\psi(s_1, y) = y$ and $\psi(s_2, y) = 0$.

$6$. A rod of rectangular cross section $[s_1, s_2] \times [b_1, b_2]$, consisting of three rectangular rods bonded together. Let the shear modulus $\mu$ be a piecewise constant function of $z$.

$$\mu = \mu_0, \quad \mu = \mu_1, \quad \mu = \mu_2$$

and $\psi = \psi(s, y)$. The integration constants are fixed by the conditions $\psi(s_1, y) = y$ and $\psi(s_2, y) = 0$.

The rod discussed in Sect. 6 has the following flexural rigidities:

$$\frac{\mu d^4}{E} \left[ \left( 1 + \psi(x, y) \right) \right] \left( 1 - \psi(x, y) \right)$$

The flexural and shear energy in the direction of the $z$ axis are, respectively, $\mu d^4/12$ and $\mu d^4/12$. Their ratio is characterized by the quantity $\eta = \frac{\mu d^4}{\mu d^4}$.

If the stress state is such that $\eta = \frac{\mu d^4}{\mu d^4}$, then the energy of transverse shear has the same order of magnitude as the energy of flexure, and the transverse shear energy must be included in the first approximation. Formulas (4.1) show that $\eta$ becomes small when the shear modulus have a large gradient. For example, the effect becomes substantial when $s = \frac{\mu d^4}{\mu d^4}$ and for the stress states with $\eta = \frac{\mu d^4}{\mu d^4}$.

REFERENCES

2. BERNOUILLI V.I., On the energy of an elastic rod, PM Vol. 45, No. 4, 1981.


Translated by L.K.