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THE PROBLEM OF AVERAGING RANDOM STRUCTURES IN TERMS OF DISTRIBUTION FUNCTIONS

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A new approach to solving averaging problems for micro-inhomogeneous continua, based on a restatement of the problem in terms of distribution functions, is described. Problems having a variational structure are considered. It is shown that, in terms of distribution functions, they reduce to the problem of minimizing a linear functional, having the meaning of the expectation value of the energy, in a set of distribution functions which is distinguished by an infinite number of linear constants. These constraints express certain matching conditions and contain multipoint distribution functions of the random characteristics of the medium. The constraints form an unlinked chain, the break of which at the n-th step contains only n-point distribution functions. In view of this, a sequence of approximate problems arises.

1. Formulation of the problem.

While there are several ways of studying composite materials, they all leave to one side the question of the distribution functions of the microfields, say the distribution functions of the stresses in the polycrystals. The microfields in essence remain unknown since, to know a random field means to know the family of its distribution functions. In view of this, it becomes necessary to state the problem of the behaviour of the material in such a way that it is a problem of finding the family of distribution functions which characterize the material by its microfields. As an example of a physical theory which is constructed in these terms, the theory of rarefied gases may be mentioned, in which, as in theory of composite materials, there are equations for the mean characteristics (the equations of gas dynamics), while to study the microfields we have to turn to the equation for the one-point distribution function (Boltzmann's equation). Though we can hardly count on such a simple situation in the theory of materials, there must undoubtedly be some analogies with Boltzmann's theory. Below, we state exactly the problem of the behaviour of composite materials in terms of distribution functions. A sequence of problems then arises which recalls, if we continue the association with the theory of gases, the chain of Bogolyubov-Born-Green-Kirkwood-Ivon equations.

We shall start from a statement of the material behaviour problems in terms of the realization of random fields.

A clear mathematical statement of such problems is obtained after expressing, as in /1/ for media with a periodic structure, the idea of the asymptotic nature of the averaging problem (see also /2, 3/ and the references cited there). The ratio of the scale of the inhomogeneity (the cell step in the periodic case) to the characteristic scale of the problem is a small parameter. The required functions are regarded as functions of fast and slow variables. The fast variables vary in a cell, while the slow variables "number" the cells. To a first approximation, to find the dependence on the fast variables, we have the so-called problem in a cell, while the dependence on the slow variables is found from the averaged equations. The coefficients of the averaged equations (and the form of these equations in the case of non-linear problems) are found from the solution of the cell problem. Thus solution of the averaging problem reduces to solution of the cell problem.

The asymptotic approach has been extended to the case of almost periodic and random media /4, 5/. The analogue of the cell problem then proved to be the problem of the behaviour of a continuous medium in an infinite space for a typical realization of the random characteristics of the medium. Its statement proved to be a very important step in understanding the averaging problem and enabled the well-known heuristic relations to be established and some new general facts to be proved /4-10/. We shall follow /7, 8/ when stating this problem.

We consider a continuous medium defined by the Lagrangian \( \Lambda \), which depends on the physical characteristic \( a \) of the medium and on the derivatives \( u_i \) of the required function \( u \) with respect to the space coordinates: \( \Lambda = \Lambda (a, u_i) \). In the context of applications, the space is assumed to be three-dimensional, and the subscript \( i \) covers the values 1, 2, 3; but the dimensionality is not important in what follows. If there are several required functions and

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characteristics of the medium, then \( u \) and \( a \) denote the respective sets of quantities. The Lagrangian \( \Lambda \) can signify the internal energy of an elastic body (\( a \) is the modulus of elasticity, and \( u \) are the components of the displacement vector), the dissipation in a heat-conducting body (\( a \) is the thermal conductivity, and \( u \) is the temperature), or the dissipation in Stokes' flow of a viscous fluid (\( a \) is the coefficient of viscosity, and \( u \) are the velocity components) etc.

We introduce the auxiliary space \( R \) of fast variables \( y_i \), the set of which is denoted by \( y \). We assume that the system of coordinates \( y_i \) in \( R \) is Cartesian. Given any function \( \psi(y) \), we define its mean over the space by

\[
\langle \psi(y) \rangle = \lim_{\lambda \to \infty} \frac{1}{\lambda^d} \int_{-\lambda/2}^{\lambda/2} \cdots \int_{-\lambda/2}^{\lambda/2} \psi(y) d^d y
\]

It is understood that the limits and integrals encountered throughout exist.

Every sample of the continuous medium is given by a field of characteristics \( a(y) \) (see /4, 5, 7/ for details). Given any function \( \psi(y) \) and numerical parameters \( v_i \), we can define a number \( \langle \Lambda (a(y), v_i + \psi_i(y)) \rangle \), where the vertical bar in the subscripts denotes differentiation with respect to \( y_i \): \( \psi_i = \partial \psi/\partial y_i \). The Lagrangian is lower-bounded (it can be assumed without loss of generality that \( \Lambda \geq 0 \)), so that the variational problem \( \inf_{\psi} \langle \Lambda (a(y), v_i + \psi_i) \rangle \) is meaningful. It is non-trivial if we impose on \( \psi \) constraints, excluding the case when the \( \psi_i \) are identically constant, or otherwise, e.g., for strictly convex non-negative functions \( \Lambda \), \( \Lambda (a(y), 0) = 0 \), the minimum is reached at the point \( v_i + \psi_i = 0 \). In particular, it can be assumed that \( \psi(y) \) is bounded at infinity. The problem then arising is a cell problem for a random structure. The macro-Lagrangian \( \bar{\Lambda} \) depends on the macrovariables \( u_i \), which signify the derivatives of the macrofields \( u \), and is given by /7, 11, 4, 5/\n
\[
\bar{\Lambda}(v_i) = \inf_{\psi_i} \langle \Lambda (a(y), v_i + \psi_i(y)) \rangle
\]

where the infimum is sought in the set of functions \( \psi(y) \) bounded at infinity. The \( v_i \) in (1.1) are regarded as parameters. If \( a(y) \) is a periodic function, it can be shown /7/ that the minimum in (1.1) need only be sought among periodic functions \( \psi(y) \) and (1.1) becomes the expression obtained in /11/ for periodic structures: the angle brackets are then understood as signifying the average over a cell.

A simple example is the problem for a heat-conducting body at rest. Here, \( \Lambda (a, u_i) = \sum \frac{1}{2} a_{ij} u_i u_j \) (summation is carried out over repeated subscripts or superscripts), \( a_{ij} \) are the thermal conductivities, while, since the problem is linear, the macro-Lagrangian \( \bar{\Lambda} \) is quadratic in the gradients of the mean temperature \( \bar{\psi} \): \( \bar{\Lambda} = \frac{1}{2} a_{ij} \bar{\psi}_i \bar{\psi}_j \). \( a_{ij} \) are the effective thermal conductivities, which are found from the variational problem

\[
\sum_{v_i} a_{ij} v_i v_j = \inf_{\psi_i} \langle \sum_{v_i} a_{ij} (v_i + \psi_i) (v_j + \psi_j) \rangle
\]

By choosing different samples of the random field \( a(y) \) we can in general obtain different values of the macro-Lagrangian \( \bar{\Lambda} \). For problem (1.2), the following important assertion was proved in /5/: if \( a(y) \) is a homogeneous ergodic random field, then the random fields \( \psi_i \), realizing the minimum in (1.2), are also homogeneous and ergodic, while \( \bar{\Lambda} \) is independent of the choice of sample of the field \( a(y) \). We assume henceforth that this assertion holds for all the problems considered.

Since \( \psi(y) \) is bounded at infinity, we have /7/

\[
\langle \psi_i(y) \rangle = 0
\]

For homogeneous ergodic fields it can happen that the variational problem (1.1), in which the condition that \( \psi(y) \) be bounded at infinity is replaced by condition (1.3), has the same solution as the initial problem. Later, for clarity, we consider problem (1.1) under condition (1.3), which is easier to take into account, and it will become clear how to replace this by the condition that \( \psi(y) \) be bounded.

The aim of our subsequent constructions is to restate problem (1.1) in terms of distribution functions.

2. Some heuristic considerations.

We introduce the density \( f(a, \psi) \) of the joint distribution of fields \( a(y) \) and \( \psi(y) \) (we shall retain a bar in the subscript when we wish to emphasize specially that \( \psi_i \) is a potential vector field). Since the random fields \( a, \psi_i \) are homogeneous, the function \( f \) is independent of \( u \), and since they are ergodic, the minimized functional can be written as
\[ \langle \Lambda(a(y), v_1 + \psi_i(y)) \rangle = \int \Lambda(a, v_1 + \psi_i) f(a, \psi_i) \, da \, dv_\psi \]  \hspace{1cm} (2.1) \]

where \( da \) and \( dv_\psi \) are elements of integration in the spaces of variables \( a \) and \( \psi_i \).

Condition (1.3) is rewritten in terms of distribution functions as

\[ \int \psi_i f(a, \psi_i) \, da \, dv_\psi = 0 \]  \hspace{1cm} (2.2) \]

The function \( f(a, \psi_i) \) also satisfies the natural conditions

\[ \int f(a, \psi_i) \, dv_\psi = f_0(a), \quad f(a, \psi_i) \geq 0 \]  \hspace{1cm} (2.3) \]

where \( f_0(a) \) is the given one-point distribution function of the homogeneous field \( a(y) \).

Note that (2.1) is a linear functional, call it \( I(f) \), of the distribution function.

Relations (2.2) and (2.3) form a system of linear constraints on \( f \). It is obvious that the search for the minimum over the fields \( \psi(y) \) is equivalent to minimizing \( I(f) \) with respect to the distribution functions of this field, and we can write

\[ \bar{X}(v_1) = \inf_f I(f) \]  \hspace{1cm} (2.4) \]

We shall seek the minimum in (2.4) with respect to all functions \( f \) which satisfy constraints (2.2), (2.3). The problem can be solved explicitly. For simplicity, we consider the case of an isotropic \( (a^1 = a^0) \) heat-conducting medium. Introducing the Lagrange multiplier \( \lambda_i \) for condition (2.2), we arrive at the variational problem

\[ \inf_f \int \frac{1}{2} a(v_1 + \psi_i)(v_1 + \psi_i) - \lambda_i \psi_i f(a, \psi_i) \, da \, dv_\psi \]  \hspace{1cm} (2.5) \]

where the minimum is sought with respect to all functions \( f \) which satisfy conditions (2.3).

The structure of problem (2.5) is as follows: it is required to find the minimum of the functional

\[ \int \Lambda(t, x) f(t, x) \, dt \, dx \]

with respect to non-negative functions \( f \) with a given value for each \( t \) of the integral of \( f \) with respect to \( x \). Denote the integral by \( f_0(t) \). It is clear how the minimizing function \( f^* \) is constructed: for each \( t \) we have to find the point \( x^*(t) \) at which the function \( \Lambda(t, x) \) has its minimum value with respect to \( x \), and concentrate \( f^*(t, x) \) at this point: \( f^*(t, x) = f_0(t) \delta((x - x^*(t))(0) \) is the Dirac delta function).

The solution of problem (2.5) thus reduces to minimizing the expression in the square brackets in (2.5) with respect to \( \psi_i \). The minimizing element \( \psi_i^* \) is given by the equation

\[ a(v_1 + \psi_i^*) = \lambda_i, \]

so that \( f^*(a, \psi_i) = f_0(a) \delta(\psi_i - a^0 v_1 + v_i) \). The Lagrange multiplier \( \lambda_i \) is found from condition (2.2); \( \lambda_i = (Ma^0)^2 v_i \) (\( M \) denotes the expectation value). Substituting \( f^* \) into (2.4), we find the minimum value of the functional, \( \frac{1}{2} (Ma^0)^2 v_i v_i \). We have thus arrived at Rice's expression, which gives a lower bound for the effective heat conduction, though it is not exactly true for any macro-isotropic structure (except for the trivial case \( a_y \equiv \text{const} \)). It can be assumed that this is true for two reasons: first, instead of using all the information about the field \( a(y) \), we only used its one-point distribution function; and second, we did not take into account that \( \psi_i \) are the derivatives of a function \( \psi \), which in general can lead to extra constraints on function \( f(a, \psi_i) \).

We first consider the statement of these extra constraints, which is of independent interest.

3. Constraints on the distribution function.

Let \( f_n = f(y^{(n)}, \psi^{(n)}, \psi_i^{(n)}, \ldots; y^{(m)}, \psi^{(m)}, \psi_i^{(m)}) \) be the \( n \)-point distribution function of random fields \( \psi, \psi_i \). The family of distribution functions \( (n = 1, 2, \ldots) \) satisfies the following matching conditions: for all \( n \)

the function \( f_n \) is symmetric with respect to any pair of arguments, \( f_n \geq 0 \) \hspace{1cm} (3.1) \]

\[ \int f_n \, d\psi^{(m)} \, dv_\psi^{(m)} = 1 \]  \hspace{1cm} (3.2) \]

The arguments \( f_n \) in (3.1) are understood to be the quantities distinguished in the writing of \( f_n \) by a point with a comma. We have the following theorem.

Theorem 1. In order for the family of distribution functions of the fields \( \psi, \psi_i \) to be a family of distribution functions of the random field \( \psi(y) \) and its derivatives \( \psi_i(y) \), it is necessary that the \( f_n \) satisfy the matching conditions (3.1) and (3.2), while for \( f_2 = f(y, \psi, \psi_i; y', \psi', \psi_i') \) we must have the equation
\[
\frac{\partial^2}{\partial y^i \partial y^j} \int f_2 dv_{\Psi'} dv_\Psi + \frac{\partial}{\partial y^i} \frac{\partial}{\partial \Psi} \int f_2 \Psi_i dv_{\Psi'} dv_\Psi + \frac{\partial}{\partial y^i} \frac{\partial}{\partial \Psi} \int f_2 \Psi_i \Psi'_j dv_{\Psi'} dv_\Psi = 0
\] (3.3)

Here, \( dv_\Psi \) is a volume element in the space of variables \( \Psi' \). It is assumed that all the integrals and derivatives in (3.3) exist. If, in addition, \( f_2 \to 0 \) as \( \Psi \to \infty, \Psi' \to \infty \) so fast that the following integrals over the sphere \( \Sigma \), or radius \( r \) in the space of variables \( \Psi (d\sigma_\Psi \) is a volume element of \( \Sigma \)), tend to zero as \( r \to \infty \):

\[
\int f_2 \Psi_i dv_{\Psi'} dv_\Psi, \quad \int f_2 \Psi_i \Psi'_j dv_{\Psi'} dv_\Psi
\]

then the conditions mentioned are also sufficient.

**Notes.** 1°. We write (3.3) under the assumption that \( f_2 \) is differentiable. Otherwise, (3.3) must be understood in the weak sense, i.e., as an integral identity.

2°. If \( \Psi \) is a vector, and not a scalar, field, then \( \partial/\partial \Psi \) in (3.3) must be understood as a divergence, e.g., the form (summation over \( \alpha, \beta \))

\[
\frac{\partial}{\partial \Psi^\alpha \partial \Psi^\beta} \int \Psi_1 \Psi'_1 f_2 dv_{\Psi'} dv_\Psi
\]

**Proof.** Necessity. Let \( \alpha (y, \Psi; y', \Psi') \) be a finite function of its arguments, and \( \Psi (y) \) a realization of a differentiable random field \( \phi \). We construct the function \( \tilde{\beta} (y, y') = \alpha (y, \Psi (y); y', \Psi (y')) \). Since \( \alpha \) is finite in \( y, y' \), then \( \tilde{\beta} (y, y') \) is also finite in \( y, y' \), so that

\[
\int \frac{\partial \tilde{\beta} (y, y')}{\partial y^i \partial y'^j} \, d\Psi^i \, d\Psi'^j = 0
\]

Writing this equation in terms of function \( \alpha \), we have

\[
\int \left( \frac{\partial \alpha}{\partial y^i} \frac{\partial \Psi (y)}{\partial y'^j} + \frac{\partial \alpha}{\partial \Psi^i} \frac{\partial \Psi (y)}{\partial y'^j} + \frac{\partial \alpha}{\partial y^i} \frac{\partial \Psi (y')}{\partial y'^j} + \frac{\partial \alpha}{\partial \Psi^i} \frac{\partial \Psi (y')}{\partial y'^j} \right) \, d\Psi^i \, d\Psi'^j = 0
\] (3.4)

Taking the expectation value of the left-hand side, we have

\[
\int \left( \frac{\partial \alpha}{\partial y^i} \frac{\partial \Psi (y)}{\partial y'^j} + \frac{\partial \alpha}{\partial \Psi^i} \frac{\partial \Psi (y)}{\partial y'^j} + \frac{\partial \alpha}{\partial y^i} \frac{\partial \Psi (y')}{\partial y'^j} + \frac{\partial \alpha}{\partial \Psi^i} \frac{\partial \Psi (y')}{\partial y'^j} \right) \Psi_i \Psi'_j f_2 \, d\Psi \, d\Psi' \, dv_\Psi \, dv_{\Psi'} \, d\Psi^i \, d\Psi'^j = 0
\] (3.5)

Since \( \alpha (y, \Psi; y', \Psi') \) is arbitrary, (3.3) follows from (3.5). The necessity of matching conditions (3.1), (3.2) is well-known, see e.g., /12/.

**Sufficiency.** Assume that we have a family of distribution functions that satisfy the matching conditions (3.1) and (3.2). Then, by Kolmogorov's theorem /12/, there are random fields \( \Psi (y, \omega), \Psi_i (y, \omega) \) (\( \omega \) is an element of probability space) whose distribution functions belong to this family. We show that, with probability one, by virtue of Eq. (3.3), the functions \( \Psi_i (y, \omega) \) are the derivatives with respect to \( y^i \) of \( \Psi (y, \omega) \). For this, it suffices to show that

\[
M \left[ \Psi(y, \omega) - \Psi(y', \omega) - \int f_i (y, \omega) \, dy^i \right]^2 = 0
\] (3.6)

where \( \Gamma \) is a contour joining the points \( y \) and \( y' \). Removing the brackets in (3.6) and taking the expectation value, we obtain

\[
B(y, y) + B(y', y') - 2B(y, y') + \int \int \, B_{ij} (z, z') \, dz^i dz^j -
\]

\[
2 \psi_i \psi' (f (y, \Psi, \Psi; z, \Psi, \Psi') - f (y', \Psi, \Psi; z, \Psi, \Psi')) \times
d\Psi \, d\Psi' \, dv_\Psi \, dv_{\Psi'} \, dz^i = 0, \quad B(y, y') = M \psi (y, \omega) \psi (y', \omega),
\]

\[
B_{ij} (y, y') = M \psi_i (y, \omega) \psi_j (y', \omega)
\] (3.7)
We now multiply (3.3) by \( \psi \psi' \) and integrate the result with respect to \( \psi \psi' \), and then integrate over the contour \( \Gamma \) with respect to \( y^j \) and \( y'^j \). It can be shown that Eq. (3.7) is then obtained (we need the condition for the integrals over \( \Sigma \), to tend to zero, in order for the terms that appear in the integration by parts to vanish). Thus, (3.6) follows from (3.3), and for (almost) every sample

\[
\psi(y, \omega) - \psi(y', \omega) - \oint_{\Gamma} \psi_1(z, \omega) dz = 0
\]

Consequently, \( \psi_1(y, \omega) \) are the derivatives with respect to \( y \) of the function \( \psi(y, \omega) \), which it was required to prove.

**Theorem 2.** If \( \psi_1 \) is the gradient of the field \( \psi \), we have for the one-point distribution function \( f(y, \psi, \psi_1) \)

\[
\frac{\partial}{\partial y^j} \int f d\psi + \frac{\partial}{\partial \psi^j} \int \psi f d\psi = 0
\]

Eq. (3.8) holds in the same way as (3.3).

We now consider the homogeneous random fields for which \( f_1 \) is independent of \( y \), while \( f_2 \) depends on the coordinates only via the difference \( \tau = y' - y \). We assume that \( f_2 \to 0 \) so fast as \( \psi \to \infty, \psi' \to \infty \), that, when integrating by parts expressions of the type \( \psi \partial f_2 / \partial \psi' \), the term outside the integral vanishes.

**Corollaries.** 1°. For homogeneous random fields

\[
\frac{\partial}{\partial \psi^a} \int f(\psi^a, \psi_1^a) \psi_1^a d\psi = 0
\]

The superscript \( a \), which numbers the set of functions \( \Psi \), is restored in (3.9) if there are several of them, in order to emphasize that Eq. (3.9) signifies the "incompressibility" for every \( i \) of the mean field of gradients in the space of variables \( \Psi^a \).

2°. The mean value of the field gradient is zero:

\[
\int f(\psi, \psi_1) \psi_1 d\psi = 0
\]

We obtain (3.10) from (3.9) by multiplying by \( \Psi^a \) and integrating by parts.

3°. For homogeneous fields \( f_1 \), we have the equation

\[
- \frac{\partial}{\partial \tau^a} \int f_1 d\psi d\psi' + \frac{\partial}{\partial \psi^a} \frac{\partial}{\partial \psi'} \int f_2 \psi_1 d\psi d\psi' = 0
\]

4°. We have the equation

\[
\sum_a \int f_2(\psi_1, \psi_1') \psi_1^a \psi_1'^a d\psi d\psi' = \frac{\partial B(\tau)}{\partial \tau^a} = \sum_a B^a(\tau) \psi^a(y + \tau)
\]

It is obtained from (3.11) by multiplying by \( \psi, \psi' \) and integrating with respect to \( \psi, \psi' \).

**Theorem 3.** The necessary and sufficient conditions for the random homogeneous field \( \psi_i(y, \omega) \) to be a field of derivatives of a random function \( \psi(y, \omega) \) are that the family of distribution functions satisfies matching and homogeneity conditions and that there is a function \( B(\tau) \) such that (3.12) holds.

The proof is similar to that of Theorem 1, the only difference being that, instead of (3.6), we establish the equation \( (\Gamma \) is any closed contour)

\[
\sum_a M(\oint_{\Gamma} \psi_1^a(z, \omega) dz)^2 = 0
\]

4. The averaging problem in terms of distribution functions. In Sect. 3 we obtained all the relations needed for restating the problem in a cell. Let \( f_n \) be the family of \( n \)-point distribution functions of random fields \( a(y) \), \( \psi_i(y) \), which satisfy the matching conditions. The field \( a(y) \) is given, so that the functions \( f_n \) in the equation

\[
\sum_a M(\oint_{\Gamma} \psi_1^a(z, \omega) dz)^2 = 0
\]
\[ \int f(y^{(1)}, a^{(1)}, \psi^{(1)}, \ldots; y^{(n)}, a^{(n)}, \psi^{(n)}) \, dv_{\psi^{(1)}} \cdots dv_{\psi^{(n)}} = \int f(y^{(1)}, a^{(1)}, \ldots; y^{(n)}, a^{(n)}) \]  

are known.

We consider the problem of minimizing the functional \( I(f) \) in the set of families of distribution functions which is distinguished by the constraints (2.2), (3.12), and (4.1), and we assume that the lower bound of \( I(f) \) is reached in this set.

**Theorem 4.** Eq. (2.4) holds, in which \( \inf \) is sought over all the distribution functions that satisfy the matching conditions and conditions (2.2), (3.12), and (4.1).

**Proof.** Let \( \psi \) be the minimizing element of the cell problem. Let \( f^* \) denote the family of distribution functions of the fields \( a(y), \partial \psi(y)/\partial y \). This family satisfies the matching conditions and (2.2), (3.12), (4.1), so that

\[ \inf_I I(f) \leq I(f^*) = \Lambda(v_i) \]  

(4.2)

Now let \( f^* \) be the minimizing family of distribution functions in our problem. Since it satisfies the matching conditions, there exist by Kolmogorov's theorem random fields \( \psi_i(y, \omega) \) for which these functions are distribution functions. Since Eq. (3.12) holds, there is a random field \( \psi(y, \omega) \) for which \( \psi_i = \partial \psi(y, \omega)/\partial y \). Moreover, \( M(\Lambda(a, v_i + \psi)) = <\Lambda(a, v_i + \psi)> \leq \infty \). On taking \( \psi(y, \omega) \) as the test function of the cell problem, we obtain

\[ \Lambda(v_i) \leq <\Lambda(a, v_i + \psi)> = M(\Lambda(a, v_i + \psi)) = \inf_I I(f) \]  

(4.3)

From inequalities (4.2) and (4.3) we obtain the theorem.

**Notes.** 3° Our variational problem is a problem of minimizing a functional \( I(f) \) which is linear in \( f \), in a set of functions which satisfy linear constraints, i.e., in this case it is a linear programming problem. Note that our theorem reduces both linear and non-linear (with non-quadratic Lagrangians) problems to a linear programming problem.

4° Without dwelling on the fact in detail, note that the cell problem for a periodic structure also reduces to a linear programming problem. Here, by the probability of the values of field \( \psi_i \) lying in a domain \( A \) of the space of variables \( \psi_i \), we have to understand the volume part of the cell in which \( \psi_i \) takes values from \( A \). Relation (2.1), then refers to finding the Lebesque integral over the cell. The interpretation of \( \Psi \)-point distribution functions is similar.

5. The sequence of approximations.

The above statement of the averaging problem contains an infinite number of constraints. In this connection a sequence of approximation problems arises, in each of which there is only a finite number of constraints. In the problem of the first approximation, only one-point distribution functions participate, in the second, only two-point functions, etc. In each successive problem, more and more detailed characteristics of the random field \( a(y, \omega) \) are taken into account.

Let \( I_{(n)} \) be the lower bound of the functional \( I(f) \) in the set distinguished by the constraints of all the \( k \)-point distribution functions with \( k \leq n \).

Consider the possibility of the equation

\[ \lim_{n \to \infty} I_{n} = \Lambda(v_i) \]  

(5.1)

As \( n \) increases, the set of admissible functions contracts, so that \( I_{(n)} \leq I_{(n+1)} \). By (4.2), the sequence \( I_{(n)} \) is upper bounded by \( \Lambda(v_i) \). It therefore has a limit. Assume that it is less than \( \Lambda(v_i) \). Let \( I_{(n)} \) be the minimizing element of the \( n \)-th approximation problem. If there is a field \( \psi_{(n)}(y) \) and a field \( \psi_{(n)}(y) \) such that \( I_{(n)} \) is the joint distribution of \( \psi_{(n)}(y) \) and \( \psi_{(n)}(y) \), then \( I_{(n)} = <\Lambda(a_{(n)}(y), v_i + \psi_{(n)}(y))> \). As \( n \to \infty \), the fields \( a_{(n)}(y) \) and \( \psi_{(n)}(y) \) approximate in their distributions, and for sufficiently large \( n \) we have \( I_{(n)} = <\Lambda(a(y), v_i + \psi_{(n)}(y))> \), which leads to a contradiction with our assumption.

If we introduce into the space of one-point distribution functions the "energy norm" (2.1) (recall that \( f \gg 0, \Lambda \gg 0 \)), then Eq. (5.1) can be interpreted as convergence in the energy norm of one-point distribution functions of the approximate problems to the one-point function of the initial problem. The question of the convergence of many-point distribution functions remains open.

The first three approximations are of greatest interest; higher order approximations are scarcely necessary.
6. The dual problem. The approximation error can be efficiently monitored by using the dual problem. In terms of realizations, it can be stated as

$$
\bar{\Lambda}(v_i) = \sup_{p^i} \left( \langle p^i \rangle v_i - \langle \Lambda^*(a(y), p^i(y)) \rangle \right)
$$

where the sup is sought over all constraints at infinity on the vector fields $p^i(y)$ which satisfy the constraint

$$
p^i_{\|i}(y) = 0
$$

and $\Lambda^*(a, p^i)$ is the Young-Fenchel transformation of $\Lambda(a, u_i)$ with respect to the variables $u_i$:

$$
\Lambda^*(a, p^i) = \sup_{u_i} (p^i u_i - \Lambda(a, u_i))
$$

The general solution of Eq. (6.2) can be written as

$$
p^i = \epsilon^{ijk} \psi_{jk}
$$

where $\epsilon^{ijk}$ are Levi Civita symbols. The dual can therefore be written in the same form as the initial problem, with the integrand $L(a, \psi_i, v_i) = \epsilon^{ijk} \psi_{ijk} v_i - \Lambda^*(a(y), \epsilon^{ijk} \psi_{jk})$. The statement of the dual problem in terms of distribution functions is the same as for the initial problem, with $\Lambda$ replaced by $L$, inf by sup, and $\psi$ by $\psi_i$.

Let $J_{(n)}$ denote the upper bound of the functional

$$
J = \int L(a, \psi_i, v_i) f(a, \psi_i) da dv
$$
in the set distinguished by the constraints on the $k$-point distribution functions, $k = 1, \ldots, n$. In the same way as in Sect.5, we can show that

$$
\lim_{n \to \infty} J_{(n)} = \bar{\Lambda}(v_i)
$$

The sequence $J_{(n)}$ is decreasing: $J_{(n+1)} \leq J_{(n)}$, and moreover $\bar{\Lambda}_{(n)} \leq \bar{\Lambda}(v_i) \leq J_{(n)}$. These inequalities can provide two-sided bounds on the macro-Lagrangian.

7. Approximations and extremality with respect to microstructures.

We specify an $n$-point distribution function of the field $a(y)$ and denote by $A_n$ the set of fields $a(y)$ which have this distribution function. Since the macro-Lagrangian $\bar{\Lambda}(v_i)$ depends on the choice of field $a(y)$, we note this by assigning to it the subscript $a$: $\bar{\Lambda} = \bar{\Lambda}_a(v_i)$. We state without proof the following property of $n$-th approximation problems, which explains their significance:

$$
I_{(n)} = \inf_{a \in A_n} \bar{\Lambda}_a(v_i), \quad J_{(n)} = \sup_{a \in A_n} \bar{\Lambda}(v_i)
$$

It can be shown that all the well-known results of averaging theory, which contain one-point distribution functions (Dykhe's formula, the Lur'e-Cherkayev estimates, etc.), can be extracted from the first approximation problem. The second approximation cannot be studied analytically; it seems only to be open to numerical methods.

REFERENCES

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