THEORY OF ANISOTROPIC THIN-WALLED CLOSED-CROSS-SECTION BEAMS

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Abstract—A variationally and asymptotically consistent theory is developed in order to derive the governing equations of anisotropic thin-walled beams with closed sections. The theory is based on an asymptotic analysis of two-dimensional shell theory. Closed-form expressions for the beam-stiffness coefficients, stress and displacement fields are provided. The influence of material anisotropy on the displacement field is identified. A comparison with the displacement fields obtained by other analytical developments is performed. The stiffness coefficients and static response are also compared with finite element predictions, closed-form solutions and test data.

INTRODUCTION

Elastically tailored composite designs are being used to achieve favorable deformation behavior under a given loading environment. Coupling between deformation modes such as extension-twist or bending-twist is created by an appropriate selection of fiber orientation, stacking sequence and materials. The fundamental mechanism producing elastic tailoring in composite beams is a result of their anisotropy. Several theories have been developed for the analysis of thin-walled anisotropic beams. A review is provided in Hodges (1990). A basic element in the analytical modeling development is the derivation of the effective stiffness coefficients and governing equations which allows the three-dimensional (3D) state of stress to be recovered from a one-dimensional (1D) beam formulation. For isotropic or orthotropic materials this is a classical problem, which is considered in a number of text books such as Timoshenko and Goodier (1951), Sokolinikov (1956), Washizu (1968), Crandall et al. (1978), Wempner (1981), Gjelsvik (1981), Libai and Simmonds (1988) and Megson (1990).

For generally anisotropic materials a number of 1D theories have been developed by Reissner and Tsai (1972), Mansfield and Sobey (1979), Rehfield (1985), Libove (1988), Rehfield and Atiglan (1989) and Smith and Chopra (1990, 1991). A discussion of these works is provided in the comparison section of this paper.

The objective of this work is to develop a consistent theory for thin-walled beams made of anisotropic materials. The theory is an asymptotically correct first-order approximation. The accuracy of previously developed theories is assessed by comparing the resulting displacement fields. A comparison of the stiffness coefficients and static response with finite element predictions, closed-form solutions and test data is also performed.

A detailed derivation of the theory is presented first. This is followed by a summary of the governing equations. Finally, a comparison of results obtained with previously developed theories is provided.

DEVELOPMENT OF THE ANALYTICAL MODEL

Coordinate systems

Consider the slender thin-walled elastic cylindrical shell shown in Fig. 1. The length of the shell is denoted by \( L \), its thickness by \( h \), the radius of curvature of the middle surface by \( R \) and the maximum cross-sectional dimension by \( d \). It is assumed that

\[
d \ll L \quad h \ll d \quad h \ll R.
\]
The shell is loaded by external forces applied to the lateral surfaces and at the ends. It is assumed that the variation of the external forces and material properties over distances of order $d$ in the axial direction and over distances of order $h$ in the circumferential direction, is small. The material is anisotropic and its properties can vary circumferentially and in the direction normal to the middle surface as well.

It is convenient to consider simultaneously two coordinate systems for the description of the state of stress in thin-walled beams. The first is the Cartesian system $x$, $y$ and $z$ shown in Fig. 1. The axial coordinate is $x$ while $y$ and $z$ are associated with the beam cross-section. The second coordinate system is the curvilinear system $x$, $s$ and $\xi$ shown in Fig. 2. The circumferential coordinate $s$ is measured along the tangent to the middle surface in a counter-clockwise direction, whereas $\xi$ is measured along the normal to the middle surface. A number of relationships have a simpler form when expressed in terms of curvilinear coordinates. A relationship between the two coordinate systems can be established as follows.

Define the positive vector $\mathbf{r}$ of the shell middle surface as

$$\mathbf{r} = x\hat{i}_x + y(x)\hat{i}_y + z(s)\hat{i}_z$$

where $\hat{i}_x$, $\hat{i}_y$, $\hat{i}_z$ are unit vectors associated with the Cartesian coordinate system $x$, $y$ and $z$. Equations $y = y(s)$ and $z = z(s)$ define the closed contour $\Gamma$ in the $y$, $z$ plane. The normal vector to the middle surface, $\mathbf{n}$ has two nonzero components

$$\mathbf{n} = n_x(s)\hat{i}_y + n_z(s)\hat{i}_z.$$  \hspace{1cm} (2)

The position vector $\mathbf{R}$ of an arbitrary material point can be written in the form

$$\mathbf{R} = \mathbf{r} + s\mathbf{n}.$$  \hspace{1cm} (3)
Equations (2) and (3) establish the relations between the Cartesian coordinates \( x, y, z \) and the curvilinear coordinates \( s, \xi, \zeta \). The coordinate \( \xi \) lies within the limits

\[
-\frac{h(s)}{2} \leq \xi \leq \frac{h(s)}{2}.
\]

The shell thickness varies along the circumferential direction and is denoted by \( h(s) \).

The tangent vector \( \mathbf{t} \), the normal vector \( \mathbf{n} \) and the projection of the position vector \( \mathbf{r} \) on \( \mathbf{t} \) and \( \mathbf{n} \) are expressed in terms of the Cartesian and curvilinear coordinates as

\[
\mathbf{t} = \frac{dr}{ds} = \frac{dy}{ds} i_x + \frac{dz}{ds} i_z
\]

\[
\mathbf{n} = \mathbf{t} \times \mathbf{i}_z = \frac{dz}{ds} i_y - \frac{dy}{ds} i_z
\]

\[
r_t = \mathbf{r} \cdot \mathbf{t} = y \frac{dz}{ds} + z \frac{dy}{ds}
\]

\[
r_n = \mathbf{r} \cdot \mathbf{n} = y \frac{dz}{ds} - z \frac{dy}{ds}.
\]

An asymptotical analysis is used to model the slender thin-walled shell as a beam with effective stiffnesses. The method follows an iterative process. The displacement function corresponding to the zeroth-order approximation is obtained first by keeping the leading-order terms in the energy functional. A set of successive corrections is added to the displacement function and the associated energy functional is determined. Corrections generating terms of the same order as previously obtained in the energy functional are kept. The process is terminated when the new contributions do not generate any additional terms of the same order as previously obtained.

**Shell energy functional**

Consider in a 3D space the prismatic shell shown in Fig. 2. A curvilinear frame \( x, s \) and \( \xi \) is associated with the undeformed shell configuration. Values 1, 2 and 3 denoting \( x, s \) and \( \xi \), respectively, are assigned to the curvilinear frame. Throughout this section, Latin superscripts (or subscripts) run from 1 to 3, while Greek superscripts (or subscripts) run from 1 to 2, unless otherwise stated.

The energy density of a 3D elastic body is a quadratic form of the strains

\[
U = \frac{1}{2} E^{ijkl} e_{ij} e_{kl}.
\]

The material properties are expressed by the Hookean tensor \( E^{ijkl} \). Following classical shell formulation (Koiter, 1959; Sanders, 1959), the through-the-thickness stress components \( \sigma^{13} \) are considerably smaller than the remaining components \( \sigma^{\alpha \beta} \), therefore

\[
\sigma^{13} = 0.
\]

The strains can be written as

\[
e_{\alpha \beta} = \gamma_{\alpha \beta} + \xi \rho_{\alpha \beta}
\]

where \( \gamma_{\alpha \beta} \) and \( \rho_{\alpha \beta} \) represent the in-plane strain components and the change in the shell middle surface curvatures, respectively. For a cylindrical shell these are related to the displacement variables by

\[
\gamma_{11} = \frac{\partial v_1}{\partial x}
\]

\[
2\gamma_{12} = \frac{\partial v_1}{\partial s} + \frac{\partial v_2}{\partial x}
\]

\[
\gamma_{22} = \frac{\partial v_2}{\partial s} + \frac{v}{R}
\]
\[ \rho_{11} = \frac{\partial^2 v}{\partial x^2} \]
\[ \rho_{12} = \frac{\partial^2 v}{\partial s \partial x} + \frac{1}{4R} \left( \frac{\partial v_1}{\partial s} - 3 \frac{\partial v_2}{\partial x} \right) \]
\[ \rho_{22} = \frac{\partial^2 v}{\partial s^2} - \frac{1}{R} \frac{\partial}{\partial s} \left( \frac{u_2}{R} \right) \]  

where \( v_1, v_2 \) and \( v \) represent the displacements in the axial, tangential and normal directions, respectively, as shown in Fig. 2. These are related to the displacement components in Cartesian coordinates by

\[ v_1 = u_1 \]
\[ v_2 = u_2 \frac{dy}{ds} + u_3 \frac{dz}{ds} \]
\[ v = u_2 \frac{dz}{ds} - u_3 \frac{dy}{ds} \]  

where \( u_1, u_2 \) and \( u_3 \) denote the displacements along the \( x, y \) and \( z \) coordinates, respectively.

The energy density of the 2D elastic body is obtained in terms of \( \gamma_{\alpha \beta} \) and \( \rho_{\alpha \beta} \) by the following procedure.

The 3D energy is first minimized with respect to \( \epsilon_{ij} \). This is equivalent to satisfying eqn (4). The result is

\[ \hat{U} = \min_{\epsilon_{ij}} U = \frac{1}{2} D_{ij} \epsilon_{ij} \]  

where \( D_{ij} \) represents the components of the 2D moduli. The expressions for \( D_{ij} \) are given in terms of \( E_{ijkl} \) in the Appendix.

The strain \( \epsilon_{ij} \) from eqn (5) is substituted into eqn (8). After integration of the result over the thickness \( \xi \) one obtains the energy of the shell \( \Phi \) per unit middle surface area:

\[ 2\Phi = h C_{ij} \gamma_{ij} \gamma_{ij} + \frac{h^2}{2} C_{ij} \gamma_{ij} \rho_{ij} + \frac{h^3}{12} C_{ij} \rho_{ij} \]  

where

\[ C_{ij} = \frac{1}{h} \langle D_{ij} \rangle \]
\[ C_{ij} = \frac{2}{h^2} \langle D_{ij} \rangle \]
\[ C_{ij} = \frac{12}{h^3} \langle D_{ij} \rangle \]

and a function of \( \xi \), say \( \alpha(\xi) \), between angular brackets is defined as an integral through the thickness, viz.

\[ \langle \alpha \rangle = \int_{-h(\xi)/2}^{+h(\xi)/2} \alpha(\xi) \, d\xi. \]  

For an applied external loading \( P \), the displacement field \( u_i \) determining the deformed state is the stationary point of the energy functional

\[ I = \int \Phi \, dx \, ds - \int P_i u_i \, dx \, ds. \]  

The asymptotical analysis of the shell energy functional

Zeroth-order approximation. Let \( \Delta \) and \( E \) be the order of displacements and stiffness coefficients \( C_{ijk} \), respectively. Assume that the order of the external forces is

\[ P \sim O \left( \frac{E \Delta h}{L^2} \right). \]
This assumption is shown later to be consistent with the equilibrium equations. An alternative would be to assume the order of the external force as some quantity $P$ and derive the order of the displacements as $PL^2/Eh$ from an asymptotic analysis of the energy functional.

For a thin-walled slender beam whose dimensions satisfy eqn (1), the rate of change of the displacements along the axial direction is much smaller than their rate of change along the circumferential direction. That is, for each displacement component

$$\frac{\partial v_i}{\partial x} \ll \frac{\partial v_i}{\partial s}.$$

Using eqn (6) and assuming that $d$ is of the same order as $R$, the order of magnitude of the in-plane strains and curvatures is

$$\gamma_{11} \sim O\left(\frac{\Delta}{L^2}\right)$$

$$2\gamma_{12} \sim O\left(\frac{\Delta}{d}\right)$$

$$\gamma_{22} \sim O\left(\frac{\Delta}{d^2}\right)$$

$$\rho_{11} \sim O\left(\frac{\Delta}{L^2}\right)$$

$$\rho_{12} \sim O\left(\frac{\Delta}{d^2}\right)$$

$$\rho_{22} \sim O\left(\frac{\Delta}{d^2}\right).$$

Since $\gamma_{11}$ and $\rho_{11}$ are much smaller than $\gamma_{12}$, $\gamma_{22}$ and $\rho_{12}$, $\rho_{22}$, respectively, their contribution to the elastic energy is neglected.

By keeping the leading-order terms in the strain-displacement relationships, eqn (6) can be written as

$$2\gamma_{12} = \frac{\partial v_1}{\partial s}$$

$$\gamma_{22} = \frac{\partial v_2}{\partial s} + \frac{v}{R}$$

$$\rho_{12} = \frac{1}{4R} \frac{\partial v_1}{\partial s}$$

$$\rho_{22} = \frac{\partial^2 v}{\partial s^2} - \frac{\partial}{\partial s} \left(\frac{v_2}{R}\right).$$

The order of magnitude of the shell energy per unit area and the work done by external forces is

$$\Phi \sim O\left(\frac{EA^2h}{d^2}\right)$$

$$P_1u_1 \sim O\left(\frac{EA^2h}{L^2}\right).$$
Since \( P_i u_i \ll \Phi \), the contribution of external forces is neglected. The energy functional takes the form
\[
2I = \int_0^L \left\{ 4hC^{1212}(\gamma_{12})^2 + 4hC^{1222}\gamma_{12} \gamma_{22} + hC^{2222} \gamma_{22}^2 + 4h^2 C_1^{1212} \gamma_{12} \rho_{12}
+ 2h^2 C_1^{1222} \gamma_{12} \rho_{22} + 2h^2 C_1^{2212} \gamma_{22} \rho_{12} + h^2 C_1^{2222} \gamma_{22} \rho_{22}
+ \frac{h^3}{3} C_2^{1212}(\rho_{12})^2 + \frac{h^3}{3} C_2^{1222} \rho_{12} \rho_{22} + \frac{h^3}{12} C_2^{2222}(\rho_{22})^2 \right\} ds \, dx.
\] (12)

The integrand in eqn (12) is a positive quadratic form, therefore the minimum of the functional is reached by functions \( v, v_1 \) and \( v_2 \) for which \( \gamma_{12} = \gamma_{22} = \rho_{12} = \rho_{22} = 0 \). From eqn (11), this corresponds to
\[
\frac{\partial v_1}{\partial s} = 0
\] (13)
\[
\frac{\partial v_2}{\partial s} + \frac{v}{R} = 0
\] (14)
\[
\frac{\partial^2 v}{\partial s^2} - \frac{\partial}{\partial s} \left( \frac{v_2}{R} \right) = 0.
\] (15)

The function \( v \) in eqns (14) and (15) should be single valued, i.e.
\[
\bar{\frac{\partial v}{\partial s}} = \frac{1}{l} \int_0^l \frac{\partial v}{\partial s} \, ds = 0.
\] (16)

The integral in eqn (16) is performed along the cross-sectional mid-plane closed contour \( \Gamma \). The length of the contour \( \Gamma \) is denoted by \( l \). The bar in eqn (16) and in the subsequent derivation denotes averaging along the closed contour \( \Gamma \).

Equation (13) implies that \( v_1 \) is a function of \( x \) only, i.e.
\[
v_1 = U_1(x).
\] (17)

Integrate eqn (15) to get
\[
\frac{\partial v}{\partial s} - \frac{v_2}{R} = -\varphi(x)
\] (18)
where \( \varphi(x) \) is an arbitrary function which is shown later to represent the cross-sectional rotation about the \( x \)-axis. From eqns (16) and (18), one obtains the relation between \( \varphi(x) \) and \( v_2 \):
\[
\varphi(x) = \left( \frac{v_2}{R} \right).
\]

Substitute \( v \) from eqn (14) into eqn (18) to get the following second-order differential equation for \( v_2 \):
\[
\frac{\partial}{\partial s} \left( R \frac{\partial v_2}{\partial s} \right) + \frac{v_2}{R} = \varphi(x).
\] (19)

To solve this equation, one has to recall the relations between the radius of curvature \( R \) and the components \( y(s) \) and \( z(s) \) of the position vector associated with the contour \( \Gamma \):
\[
\frac{d^2 z}{ds^2} = \frac{1}{R} \frac{dy}{ds},
\]
\[
\frac{d^2 y}{ds^2} = -\frac{1}{R} \frac{dz}{ds}.
\] (20)

It follows from eqn (20) that \( dy/ds \) and \( dz/ds \) are solutions of the homogeneous form of eqn (19) and \( v_2 = \varphi(x) \gamma_n \) is its particular solution. The general solution is therefore
Anisotropic thin-walled beams

energy is given by

\[ v_2 = U_2(x) \frac{dy}{ds} + U_3(x) \frac{dz}{ds} + \phi(x) r_n \]  

(21)

where \( U_2 \) and \( U_3 \) are arbitrary functions of \( x \). Substitute from eqn (21) into eqn (14) to get

\[ v = U_2(x) \frac{dz}{ds} - U_3(x) \frac{dy}{ds} - \phi(x) r_i. \]  

(22)

Equations (17), (21) and (22) represent the curvilinear displacement field that minimizes the zeroth-order approximation of the shell energy. Using eqn (7), the curvilinear displacement field is written in Cartesian coordinates as

\[ u_1 = U_1(x) \]

\[ u_2 = U_2(x) - z \phi(x) \]

\[ u_3 = U_3(x) + y \phi(x). \]

The variables \( U_1(x) \), \( U_2(x) \) and \( U_3(x) \) represent the average cross-sectional translation, while \( \phi(x) \) represents the cross-sectional rotation normally referred to in beam theory as the torsional rotation. This displacement field corresponds to the zeroth-order approximation and does not include bending behavior. For a centroidal coordinate system \( U_i(x) \), \( U_i(x) \) and \( U_i(x) \) and \( \phi(x) \) can be expressed as

\[ U_1(x) = \bar{u}_1 \]

\[ U_2(x) = \bar{u}_2 \]

\[ U_3(x) = \bar{u}_3 \]

\[ \phi(x) = \frac{(u \cdot t)}{r_n}. \]

First-order approximation. A first-order approximation can be constructed by replacing the displacement field in eqns (17), (21) and (22) in the form

\[ u_1 = U_1(x) + w_1(s, x) \]

\[ u_2 = U_2(x) \frac{dy}{ds} + U_3(x) \frac{dz}{ds} + \phi(x) r_n + w_2(s, x) \]

\[ v = U_2(x) \frac{dz}{ds} - U_3(x) \frac{dy}{ds} - \phi(x) r_i + w(s, x) \]

(23)

where \( w_1 \), \( w_2 \) and \( w \) can be regarded as correction functions to be determined based on their contributions to the energy functional.

Substitute eqn (23) into eqn (6) to obtain the strains and curvatures in terms of the displacement corrections

\[ \gamma_{11} = \dot{\gamma}_{11} + \frac{\partial w_1}{\partial x} \]

\[ 2 \gamma_{12} = 2 \dot{\gamma}_{12} + \frac{\partial w_2}{\partial x} + 2 \dot{\gamma}_{12} = \frac{\partial w_1}{\partial s} \]

\[ \gamma_{22} = \dot{\gamma}_{22} + \dot{\gamma}_{22}, \quad \dot{\gamma}_{22} = \frac{\partial w_2}{\partial s} + \frac{w}{R} \]

\[ \rho_{11} = \ddot{\rho}_{11} + \frac{\partial^2 w}{\partial x^2} \]

\[ \rho_{12} = \ddot{\rho}_{12} + \frac{\partial^2 w}{\partial s \partial x} - \frac{3}{4R} \frac{\partial^2 w}{\partial x^2} + \dot{\rho}_{12}, \quad \dot{\rho}_{12} = \frac{1}{4R} \frac{\partial w_1}{\partial s} \]

\[ \rho_{22} = \ddot{\rho}_{22} + \dot{\rho}_{22}, \quad \dot{\rho}_{22} = \frac{\partial^2 w}{\partial s^2} - \frac{\partial}{\partial s} \left( \frac{w_2}{R} \right) \]
where \( \dot{\gamma}_{\alpha\beta} \) and \( \dot{\rho}_{\alpha\beta} \) are the strains and curvatures corresponding to the zeroth-order approximation. These are expressed as

\[
\dot{\gamma}_{11} = U_1'(x) \\
2\dot{\gamma}_{12} = U_2'(x) \frac{dy}{ds} + U_1'(x) \frac{dz}{ds} + \varphi'(x) r_n \\
\dot{\gamma}_{22} = 0 \\
\dot{\rho}_{11} = U_2''(x) \frac{dz}{ds} - U_1''(x) \frac{dy}{ds} - \varphi''(x) r_t \\
\dot{\rho}_{12} = \frac{1}{4R} \left[ U_2'(x) \frac{dy}{ds} + U_1'(x) \frac{dz}{ds} + \varphi'(x) r_n \right] - \varphi'(x) \\
\dot{\rho}_{22} = 0.
\]

(25)

The prime in eqn (25) denotes differentiation with respect to \( x \). The order of \( w_i \) is \((\Delta d/L)\). Among the new terms introduced by the function \( w_i \), the leading ones are denoted by a superscript \(^\sim\) in eqn (24). By keeping their contribution over the other terms, the energy functional can be represented by

\[
\Phi(\dot{\gamma}_{11}, 2\dot{\gamma}_{12}, 2\dot{\gamma}_{12}, \dot{\gamma}_{22}, 0, \dot{\rho}_{12}, \dot{\rho}_{22})
\]

where terms of order \((\Delta^2 h/L^3 d)\) or smaller such as

\[
\begin{align*}
&h_{\rho_{11}} \dot{\gamma}_{12}, \ h_{\rho_{11}} \dot{\rho}_{22}, \ h^2 \dot{\rho}_{11} \dot{\rho}_{12}, \ h^2 \dot{\rho}_{11} \dot{\rho}_{22} \\
&h_{\rho_{12}} \dot{\gamma}_{12}, \ h_{\rho_{12}} \dot{\rho}_{22}, \ h^2 \dot{\rho}_{12} \dot{\rho}_{12}, \ h^2 \dot{\rho}_{12} \dot{\rho}_{22}
\end{align*}
\]

are neglected in comparison with the following terms

\[
\begin{align*}
&\dot{\gamma}_{11} \dot{\gamma}_{12}, \ \dot{\gamma}_{11} \dot{\rho}_{22}, \ \dot{\gamma}_{12} \dot{\gamma}_{12}, \ \dot{\gamma}_{12} \dot{\rho}_{22}
\end{align*}
\]

of order \((\Delta^2 / L^2)\). Similarly, the contribution of the work done by external forces, \( P_j w_i \), is neglected since its order is \((Eh(\Delta^2 / L^2)(d/L))\) in comparison with the order of the remaining terms in the energy functional \((Eh(\Delta^2 / L^2))\). Therefore in order to determine the functions \( w_i \) one has to minimize the functional

\[
\int \Phi(\dot{\gamma}_{11}, 2\dot{\gamma}_{12}, 2\dot{\gamma}_{12}, \dot{\gamma}_{22}, 0, \dot{\rho}_{12}, \dot{\rho}_{22}) \, ds.
\]

If the rigid body motion is suppressed, the solution is unique. The terms \( \dot{\rho}_{12}, \dot{\rho}_{22} \) are essential to the uniqueness of the solution; however, their contribution to the energy is of lower order \((Eh(\Delta^2 / L^2)(h/d))\) and is consequently dropped. This aspect is discussed by Berdichevsky and Misiura (1992) with regard to the accuracy of classical shell theory.

The shell energy can therefore be represented by

\[
I = \int_0^L \int_0^1 \Phi(\dot{\gamma}_{11}, 2\dot{\gamma}_{12}, 2\dot{\gamma}_{12}, \dot{\gamma}_{22}, 0, 0, 0) \, ds \, dx.
\]

(26)

It is worth noting that the bending contribution does not appear in eqn (26). That is, in the first-order approximation the shell energy corresponds to a membrane state.

The first variation of the energy functional is

\[
\delta I = \int_0^L \int_0^1 \left( \frac{\partial \Phi}{\partial (2\gamma_{12})} \delta \left( \frac{\partial w_i}{\partial s} \right) + \frac{\partial \Phi}{\partial \gamma_{22}} \delta \left( \frac{\partial w_i}{\partial s} + \frac{w}{R} \right) \right) \, ds \, dx.
\]

(27)

Equation (27) can be written in terms of the shear flow \( N_{12} \) and hoop-stress resultant \( N_{22} \) by recalling that

\[
N_{12} = \frac{\partial \Phi}{\partial (2\gamma_{12})} \quad \text{and} \quad N_{22} = \frac{\partial \Phi}{\partial \gamma_{22}}.
\]
The result is
\[
\delta I = \int_0^L \left\{ N_{12} \frac{\delta w_1}{\delta s} + N_{22} \left( \frac{\delta w_2}{\delta s} + \frac{1}{R} \delta w \right) \right\} ds \, dx.
\]

Set the first variation of the energy to zero, to obtain the following:
\[
\frac{\partial N_{12}}{\partial s} = 0
\]
\[
\frac{\partial N_{22}}{\partial s} = 0
\]
\[
\frac{N_{22}}{R} = 0
\]

which result in
\[
N_{12} = \text{constant} \quad (28)
\]
and
\[
N_{22} = 0. \quad (29)
\]

This is similar to the classical solution of constant shear flow and vanishing hoop stress resultant. By setting \(N_{22}\) to zero, the energy density is expressed in terms of \(\gamma_{11}\) and \(\gamma_{12}\) only:
\[
2\Phi_1 = \min_{\gamma_{22}} 2\Phi = A(s)(\gamma_{11})^2 + 2B(s)\gamma_{11}\gamma_{12} + C(s)(\gamma_{12})^2. \quad (30)
\]

The variables \(A(s)\), \(B(s)\) and \(C(s)\) represent the axial, coupling and shear stiffnesses, respectively. They are defined in terms of the 2D shell moduli in the Appendix.

Equation (30) indicates that, to first order, the energy density function is independent of functions \(w_2\) and \(w\). That is, the in-plane warping contribution to the shell energy is negligible. The function \(w_1\), however, can be determined from eqns (28) and (30) and by enforcing the condition on \(w_1\) to be single-valued as follows:
\[
N_{12} = \frac{\partial \Phi_1}{\partial (2\gamma_{12})} = \frac{1}{2} (B(s)\gamma_{11} + C(s)\gamma_{12}) = \text{constant}. \quad (31)
\]

Substitute the leading terms from eqns (24) and (25) into eqn (31) to get
\[
\frac{1}{2} B U_1'(x) + \frac{1}{4} C \left( U_2'(x) \frac{dy}{dx} + U_2'(x) \frac{dz}{dx} + \varphi'(x) \rho_a(s) + \frac{\partial w_1}{\partial s} \right) = \text{constant}. \quad (32)
\]

In deriving eqn (32), the term \(B(\partial w_1/\partial s)\) has been neglected in comparison with \(1/4 C(\partial w_1/\partial s)\). This is possible if \(|B|\) is less than, or of the same order of magnitude as, \(C\).

For the case when \(|B| \gg C\) additional investigation is needed. Since the elastic energy is positive definite, \(B^2 \leq AC\), and \(B\) could be greater than \(C\) only if \(A \gg C\). In practical laminated composite designs \(|B| < C\), as the shear stiffness is greater than the extension–shear coupling.

Equation (32) is a first-order ordinary differential equation in \(w_1\). The value of the constant in the right-hand side of eqn (32) can be found from the single-value condition of function \(w_1\):
\[
\left( \frac{\partial w_1}{\partial s} \right) \bigg|_{s=0} = \int_0^L \frac{\partial w_1}{\partial s} ds \bigg|_{s=0} = 0.
\]

The solution of eqn (32) is determined within an arbitrary function of \(x\). This function can be specified from various conditions. Each one yields a specific interpretation of the variable \(U_1\). For example, if \(\bar{w}_1 = 0\) the variable \(U_1 = \bar{v}_1\) according to eqn (23). The choice of these conditions does not affect the final form of the 1D beam theory and therefore will not be specified in this formulation. The result is the following simple analytical solution of eqn (32):
\[
w_1 = -y U_2'(x) - z U_1'(x) + G(s) \varphi'(x) + g_1(s) U_1'(x) \quad (33)
\]
where
\[ G(s) = \int_0^s \left[ \frac{2A_e}{l}\varsigma c(\tau) - r_n(\tau) \right] d\tau \]
\[ g_1(s) = \int_0^s \left[ b(\tau) - \frac{\theta}{\varsigma} c(\tau) \right] d\tau \]
\[ b(s) = -2 \frac{B_1(s)}{C(s)} \quad c(s) = \frac{1}{C(s)} \quad A_e = \frac{l}{2} r_n. \quad (34) \]

The area enclosed by contour \( \Gamma \) is denoted by \( A_e \) in eqn (34).

The displacement field corresponding to the first correction is obtained by substituting eqn (33) into (23) and dropping \( w_2 \) and \( w \) since their contribution to the shell energy is negligible compared to that of \( w_1 \). The result referred to as the first-order approximation is given by
\[ v_1 = U_1(x) - y(x)U_2(x) - z(x)U_3(x) + G(s)\varphi'(x) + g_1(s)U_1'(x) \]
\[ v_2 = U_2(x) \frac{dy}{ds} + U_3(x) \frac{dz}{ds} + \varphi(x)r_n \]
\[ v = U_3(x) \frac{dz}{ds} - U_2(x) \frac{dy}{ds} - \varphi(x)r_t. \]

**Displacement field.** The displacement field corresponding to the next correction is found in the same way. A third correction can also be performed. However, subsequent corrections yield only smaller terms, as shown in Badir (1992), and the displacement field converges to the following expression:
\[ v_1 = U_1(x) - y(x)U_2(x) - z(x)U_3(x) + G(s)\varphi'(x) \]
\[ + g_1(s)U_1'(x) + g_2(s)U_2'(x) + g_3(s)U_3'(x) \]
\[ v_2 = U_2(x) \frac{dy}{ds} + U_3(x) \frac{dz}{ds} + \varphi(x)r_n \]
\[ v = U_3(x) \frac{dz}{ds} - U_2(x) \frac{dy}{ds} - \varphi(x)r_t. \quad (35) \]

where
\[ g_2(s) = -\int_0^s \left[ b(\tau)\varphi(\tau) - \frac{\theta y}{\varsigma} c(\tau) \right] d\tau \]
\[ g_3(s) = -\int_0^s \left[ b(\tau)\varphi(\tau) - \frac{\theta z}{\varsigma} c(\tau) \right] d\tau. \quad (36) \]

It is seen from expressions (34) and (36) that \( G(s) \), \( g_1(s) \), \( g_2(s) \) and \( g_3(s) \) are single-valued functions, that is
\[ G(0) = G(l) = g_1(0) = g_1(l) = g_2(0) = g_2(l) = g_3(0) = g_3(l) = 0. \]

The expressions for the displacements \( v_2 \), \( v \) and the first four terms in \( v_1 \) are analogous to the classical theory of extension, bending and torsion of beams. The additional terms \( g_1(s)U_1' \), \( g_2(s)U_2' \) and \( g_3(s)U_3' \) in the expression for \( v_1 \) in eqn (35) represent warping due to axial strain and bending. These new terms emerge naturally in addition to the classical torsional-related warping \( G(s)\varphi' \). They are strongly influenced by the material's anisotropy, and vanish for materials that are either orthotropic or whose properties are antisymmetric relative to the shell middle surface. These out-of-plane warping functions were first derived by Armanios et al. (1991) for laminated composites.
The contribution of out-of-plane warping was considered recently by Kosmatka (1991). Local in-plane deformations and out-of-plane warping of the cross-section were expressed in terms of unknown functions. These functions were assumed to be proportional to the axial strain, bending curvature and twist rate within the cross-section and were determined using a finite element modeling. In the present formulation, the out-of-plane warping is shown to be proportional to the axial strain, bending curvature and torsion twist rate. The functions associated with each physical behavior are expressed in closed form by \( g_1(s) \) for the axial strain, \( g_2(s) \) and \( g_3(s) \) for the bending curvatures, and \( G(s) \) for the torsion twist rate.

**Strain field.** The strain field is obtained by substituting eqn (35) into eqn (6) and neglecting terms of smaller order in the shell energy. The result is

\[
\gamma_{11} = U'_1(x) - y(s)U'_x(x) - z(s)U'_y(x)
\]

\[
2\gamma_{12} = \frac{2A_x}{\ell c} c(s)\varphi' + \left[ b(s) - \frac{b}{\ell c} c(s) \right] U'_1
\]

\[
- \left[ b(s)\varphi(s) - \frac{b}{\ell c} c(s) \right] U'_x - \left[ b(s)z(s) - \frac{b}{\ell c} c(s) \right] U'_y
\]  

\[\gamma_{22} = 0.\]

It is worth noting that the vanishing of the hoop-stress resultant in eqn (29) and the hoop strain in eqn (37) should be interpreted as a negligible contribution relative to the other parameters. The longitudinal strain \( \gamma_{11} \) is a linear function of \( y \) and \( z \). This result was adopted as an assumption in the work of Libove (1988).

In deriving eqn (37), higher-order terms associated with \( G\varphi'' \) in the energy functional have been neglected in comparison with \( C'(\dot{\varphi}/\ell c)^2 \), as shown in Badir (1992). This is possible if the following inequalities are satisfied

\[
\frac{A}{C} \left( \frac{d}{L} \right) \ll 1 \quad \frac{B}{C} \left( \frac{d}{L} \right) \ll 1.
\]

**Constitutive relationships.** Substitute eqn (37) in the energy density, eqn (30), and integrate over \( s \) to get the energy of the 1D beam theory:

\[
I = \int_0^L \Phi_2 \, dx - \int P_i U_i \, dx \, ds
\]

where

\[
\Phi_2 = \frac{1}{2} \left[ C_{11} (U'_1)^2 + C_{22} (\varphi')^2 + C_{33} (U'_y)^2 + C_{44} (U'_x)^2 \right]
\]

\[
+ C_{12} U'_1 \varphi' + C_{13} U'_1 U'_y + C_{14} U'_1 U'_x
\]

\[
+ C_{23} \varphi' U'_y + C_{24} \varphi' U'_x + C_{34} U'_y U'_y
\]

(39)

Explicit expressions for the stiffness coefficients \( C_{ij} (i, j = 1, 4) \) are given in the Appendix. The constitutive relationships can be written in terms of stress resultants and kinematic variables by differentiating eqn (39) with respect to the associated kinematic variable or by relating the traction \( T \), torsional moment \( M_x \), and bending moments \( M_y \) and \( M_z \) to the shear flow and axial stress as follows:

\[
T = \frac{\partial \Phi_2}{\partial U'_1} = \int \sigma_{11} \, d\xi \, ds = \int N_{11} \, ds
\]

\[
M_x = \frac{\partial \Phi_2}{\partial \varphi'} = \int \sigma_{12} r_n(s) \, d\xi \, ds = \int N_{12} r_n(s) \, ds
\]

\[
M_y = \frac{\partial \Phi_2}{\partial U'_y} = -\int \sigma_{11} z \, d\xi \, ds = -\int N_{11} z(s) \, ds
\]

\[
M_z = \frac{\partial \Phi_2}{\partial U'_x} = -\int \sigma_{11} y \, d\xi \, ds = -\int N_{11} y(s) \, ds
\]
The shear flow $N_{12}$ is derived from the energy density in eqn (31) and the axial stress resultant $N_{11}$ is given by

$$N_{11} = \frac{\partial \Phi_1}{\partial y_{11}} = A(s)y_{11} + B(s)y_{12}$$

and the associated axial and shear stresses are uniform through the wall thickness.

Substitute eqn (37) into eqns (31) and (41) and use eqn (40) to get

$$\begin{pmatrix}
T \\
M_x \\
M_y \\
M_z
\end{pmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
C_{12} & C_{22} & C_{23} & C_{24} \\
C_{13} & C_{23} & C_{33} & C_{34} \\
C_{14} & C_{24} & C_{34} & C_{44}
\end{bmatrix}
\begin{pmatrix}
U_1 \\
\phi' \\
U_2'' \\
U_3'
\end{pmatrix}$$

Equilibrium equations. The equilibrium equations can be derived by substituting the displacement field in eqn (35) into the energy functional in eqn (10) and using the principle of minimum total potential energy to get

$$T' + \oint P_x \, ds = 0$$

$$M_x' + \oint (P_x y - P_y z) \, ds = 0$$

$$M_y'' + \left( \oint P_y z \, ds \right)' + \oint P_y \, ds = 0$$

$$M_z'' + \left( \oint P_z y \, ds \right)' + \oint P_z \, ds = 0$$

where $P_x$, $P_y$ and $P_z$ are surface tractions along the $x$, $y$ and $z$ directions, respectively.

One member of each of the following four pairs must be prescribed at the beam ends:

- $T$ or $U_1$, $M_x$ or $\phi$, $M_y$ or $U_2''$, and $M_z$ or $U_3'$.

SUMMARY OF GOVERNING EQUATIONS

The development presented in this work encompasses five equations. The first is the displacement field given by eqn (35). Its functional form was determined based on an asymptotical expansion of the shell energy. The associated strain field is given by eqn (37) and the stress resultants by eqns (31), (40) and (41). The fourth are the constitutive relationships in eqn (42), with the stiffness coefficients expressed as integrals of material properties and cross-sectional geometry in eqn (A4) of the Appendix. Finally, the equilibrium equations and boundary conditions are given by eqns (43) and (44), respectively.

In the present development the determination of the displacement field is essential in obtaining accurate expressions for the beam stiffnesses. A comparison of the derived displacement field with results obtained by previous investigators is presented in the following section.

COMPARISON OF DISPLACEMENT FIELDS

The pioneering work of Reissner and Tsai (1972) is based on developing an exact solution to the governing equilibrium, compatibility and constitutive relationships of shell theory. Closed as well as open cross-sections were considered. The derived constitutive relationships are similar to eqn (42). However, the authors left to the reader the derivation of the explicit expressions for the stiffness coefficients. This may be the reason why their work was overlooked. These expressions are important in identifying the parameters controlling the behavior and in performing parametric design studies. Furthermore, the explicit form of the displacement field helps to evaluate and understand predictions of other analytical and numerical models.
A number of assumptions were adopted in Reissner and Tsai’s development regarding material properties, such as neglecting the coupling between in-plane strains and curvatures which can be significant in anisotropic materials. It is important to assess the influence of these assumptions on the accuracy. This has been done in the present work by using an asymptotic expansion of the shell energy and proving that the coupling and curvature contributions to the energy are small in comparison with the in-plane contribution.

Mansfield and Sobey (1979) and Libove (1988) obtained the beam flexibilities relating the stretching, twisting and bending deformations to the applied axial load, torsional and bending moments for a special origin and axes orientation. They adopted the assumptions of a negligible hoop-stress resultant \( N_{ss} \) and a membrane state in the thin-walled beam section. Although they did not refer to the work of Reissner and Tsai (1972), their stiffnesses coincide for the special case outlined in Reissner and Tsai (1972). This special case refers to that where the classical assumptions of neglecting shear and hoop stresses and considering the shear flow to be constant is adopted. However, one has to carry out the details to show this fact.

The work of Rehfeld (1985) has been used in a number of composite applications. Rehfeld’s displacement field is of the form

\[
\begin{align*}
    u_1 &= U_1(x) - y(s)[U_2'(x) - 2\gamma_{xy}(x)] - z(s)[U_3'(x) - 2\gamma_{xz}(x)] + g(s, x) \\
    u_2 &= U_2(x) - z(s)\varphi(x) \\
    u_3 &= U_3(x) + y(s)\varphi(x)
\end{align*}
\]

where \( \gamma_{xy} \) and \( \gamma_{xz} \) are the transverse shear strains. The warping function \( g(s, x) \) is given as

\[
g(s, x) = \tilde{G}(s)\varphi'(x)
\]

with

\[
\tilde{G}(s) = 2A_6' \int_0^s r_n(\tau) \, d\tau.
\]

A comparison of the displacement fields in eqns (35) and (45) shows that the warping function in Rehfeld’s formulation comprises the torsional-related contribution but does not include explicit terms that express the bending-related warping. The torsional warping function \( G(s) \) in eqn (34) is different from the function in eqn (47). The two expressions coincide when \( c = \) constant, that is, when the wall stiffness and thickness are uniform along the cross-sectional circumference.

The torsional warping function in eqn (47) was modified by Atilgan (1989) and Rehfeld and Atilgan (1989) as

\[
\tilde{G}(s) = \int_0^s \left[ \frac{2A_6}{I_1^c} c_1 - r_n(\tau) \right] \, d\tau
\]

where

\[
c_1 = \frac{1}{A_{66}' - ([A_{16}]^2/A_{11})}
\]

and

\[
\begin{bmatrix} A_{11} & A_{16} \\ A_{16} & A_{66} \end{bmatrix} = \begin{bmatrix} A_{11} - \frac{(A_{12})^2}{A_{22}} & A_{16} - \frac{A_{12}A_{26}}{A_{22}} \\ A_{16} - \frac{A_{12}A_{26}}{A_{22}} & A_{66} - \frac{(A_{26})^2}{A_{22}} \end{bmatrix}
\]

The \( A_{ij} \) in eqn (50) are the in-plane stiffnesses of Classical Lamination Theory (Jones, 1975; Vinson and Sierakowski, 1987). They are related to the modulus tensor by

\[
A_{11} = \langle E_{1111} \rangle, \quad A_{12} = \langle E_{1122} \rangle, \quad A_{22} = \langle E_{2222} \rangle, \\
A_{16} = \langle E_{1112} \rangle, \quad A_{26} = \langle E_{2222} \rangle, \quad A_{66} = \langle E_{1212} \rangle.
\]
A comparison of the modified torsional warping function in eqn (48) and \(G(s)\) in eqn (34) shows that they coincide for laminates with no extension–shear coupling \((\langle D^{122}\rangle = 0, \text{ in eqn (A2) of the Appendix})\). For the case where the through-the-thickness contribution is neglected in eqn (A2), this reduces to \(A_{16} = A_{26} = 0\).

The warping function obtained by Smith and Chopra (1990, 1991) for composite box-beams is identical to the expression of Rehfield and Atilgan (1989) and Atilgan (1989) given in eqns (46) and (48).

An assessment of all the previous warping expressions can be made by checking whether they reduce to the exact expression for isotropic materials [see, for example, Megson (1990)]

\[
\bar{G}(s) = \int_0^s \left[ -\frac{2A_e}{l c_2} c_2 - r_n(\tau) \right] d\tau
\]

with

\[
c_2 = \frac{1}{\mu h(s)}
\]

where \(\mu\) is the shear modulus.

For isotropic materials the in-plane coupling \(b\) is zero and consequently \(\xi_1, \xi_2\) and \(\xi_3\) in eqns (34) and (36) vanish. That is, the warping is torsion-related and reduces to \(G(s)p^\circ\). Moreover, the shear parameter \(c\) is equal to \(1/4\mu h(s)\) and the expressions for \(G(s)\) and \(\bar{G}(s)\) in eqns (34) and (51) coincide.

Rehfield’s warping function in eqn (47) coincides with eqn (51) when the material properties and the thickness are uniform along the wall circumference. Atilgan's (1989), Rehfield and Atilgan's (1989) and Smith and Chopra's (1991) formulations reduce to eqn (51) for isotropic materials.

APPLICATIONS

Two special layups: the circumferentially uniform stiffness (CUS) and circumferentially asymmetric stiffness (CAS) have been considered by Atilgan (1989), Rehfield and Atilgan (1989), Hodges et al. (1989), Rehfield et al. (1990), Chandra et al. (1990) and Smith and Chopra (1990, 1991).

CUS configuration

This configuration produces extension–twist coupling. The axial, coupling and in-plane stiffnesses \(A, B\) and \(C\) given in eqn (A1) of the Appendix are constant throughout the cross-section, and hence the name circumferentially uniform stiffness (CUS) was adopted by Atilgan (1989), Rehfield and Atilgan (1989), Hodges et al. (1989) and Rehfield et al. (1990). For a box-beam, the ply layups on opposite sides are of reversed orientation, and hence the name antisymmetric configuration was adopted by Chandra et al. (1990) and Smith and Chopra (1990, 1991).

Since \(A, B\) and \(C\) are constants, the stiffness matrix in eqn (42), for a centroidal coordinate system, reduces to

\[
\begin{bmatrix}
C_{11} & C_{12} & 0 & 0 \\
C_{12} & C_{22} & 0 & 0 \\
0 & 0 & C_{33} & 0 \\
0 & 0 & 0 & C_{44}
\end{bmatrix}
\]

The nonzero stiffness coefficients are given by

\[
\begin{align*}
C_{11} &= A l \\
C_{12} &= B A_e \\
C_{22} &= \frac{C}{l} A_e^2
\end{align*}
\]
Anisotropic thin-walled beams

\[ C_{33} = A \int z^2 \text{ds} - \frac{B_t^2}{C_t} \int z^2 \text{ds} \]
\[ C_{44} = A \int y^2 \text{ds} - \frac{B_t^2}{C_t} \int y^2 \text{ds}. \]

(52)

For such a case the out-of-plane warping due to axial strain vanishes and \( g_1 \) does not affect the response.

**CAS configuration**

This configuration produces bending-twist coupling. The stiffness \( A \) is constant throughout the cross-section. For a box beam, the coupling stiffness, \( B_t \), in opposite members is of opposite sign and hence the name circumferentially asymmetric stiffness (CAS) was adopted by Atilgan (1989), Rehfield and Atilgan (1989), Hodges et al. (1989) and Rehfield et al. (1990). For a box-beam, the ply layups along the horizontal members are mirror images, and hence the name symmetric configuration was adopted by Chandra et al. (1990) and Smith and Chopra (1990, 1991). The stiffness \( C \) in opposite members is equal. The stiffness matrix, for a centroidal system of axes, reduces to

\[
\begin{bmatrix}
C_{11} & 0 & 0 & 0 \\
0 & C_{22} & C_{23} & 0 \\
0 & C_{23} & C_{33} & 0 \\
0 & 0 & 0 & C_{44}
\end{bmatrix}
\]

The nonzero stiffness coefficients are expressed by

\[ C_{11} = A l - 2 \frac{B_t^2}{C_t} d \]
\[ C_{22} = \frac{C_t}{2(d + a(C_t/C_v))} A_e^2 \]
\[ C_{23} = \frac{B_t}{2(d + a(C_t/C_v))} A_e^2 \]
\[ C_{33} = A \int z^2 \text{ds} - \frac{B_t^2}{2C_t} \left( a - \frac{A_e}{d + a(C_t/C_v)} \right) A_e \]
\[ C_{44} = A \int y^2 \text{ds} - \frac{B_t^2d^3}{6C_t}. \]

Subscripts \( t \) and \( v \) denote top and vertical members, respectively. The box width and height are denoted by \( d \) and \( a \), respectively. For the CAS configuration and with reference to the Cartesian coordinate system in Fig. 1, bending about the \( y \)-axis is coupled with torsion while extension and bending about the \( z \)-axis are decoupled.

In order to assess the accuracy of the predictions the present theory is applied to the box-beam studied by Hodges et al. (1989). The cross-sectional configuration is shown in Fig. 3 and the material properties in Table 1.
Table 1. Properties of T300/5208 graphite/epoxy

\[
\begin{align*}
E_{11} &= 21.3 \text{ Msi} \\
E_{22} &= E_{33} = 1.6 \text{ Msi} \\
G_{12} &= G_{13} = 0.9 \text{ Msi} \\
G_{13} &= 0.7 \text{ Msi} \\
\nu_{12} &= \nu_{13} = 0.28 \\
\nu_{23} &= 0.5
\end{align*}
\]

Flexibility coefficients

A comparison of the flexibility coefficients \( S_{ij} \) with the predictions from two models is provided in Table 2. The flexibility coefficients \( S_{ij} \) are obtained by inverting the \( 4 \times 4 \) matrix in eqn (42). The NABSA (Nonhomogeneous Anisotropic Beam Section Analysis) is a finite element model based on an extension of the work of Giavotto et al. (1983). In this model all possible types of warping are accounted for. The TAIL model is based on the theory of Rehfiedl (1985) while neglecting the restrained toroidal warping. The predictions of the NABSA and TAIL models are provided by Hodges et al. (1989). The percentage differences appearing in Table 2 are relative to the NABSA predictions. The present theory is in good agreement with NABSA. Its predictions show a difference ranging from +0.7 to +3.6% while those based on Rehfiedl’s theory (1985) range from +3.6 to -18.4%.

<table>
<thead>
<tr>
<th>Flexibility</th>
<th>NABSA</th>
<th>Present</th>
<th>% Diff.</th>
<th>TAIL</th>
<th>% Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_{11} \times 10^6 )</td>
<td>0.143883</td>
<td>0.14491</td>
<td>+0.7</td>
<td>0.14491</td>
<td>+0.7</td>
</tr>
<tr>
<td>( S_{22} \times 10^6 )</td>
<td>0.312145</td>
<td>0.32364</td>
<td>+3.6</td>
<td>0.32364</td>
<td>+3.6</td>
</tr>
<tr>
<td>( S_{12} \times 10^6 )</td>
<td>-0.417841</td>
<td>-0.43010</td>
<td>+2.9</td>
<td>-0.43010</td>
<td>+2.9</td>
</tr>
<tr>
<td>( S_{33} \times 10^6 )</td>
<td>0.183684</td>
<td>0.1886</td>
<td>+2.6</td>
<td>0.17294</td>
<td>-5.8</td>
</tr>
<tr>
<td>( S_{44} \times 10^6 )</td>
<td>0.614311</td>
<td>0.63429</td>
<td>+3.2</td>
<td>0.50157</td>
<td>-18.4</td>
</tr>
</tbody>
</table>

The present theory is applied to the prediction of the tip deformation in a cantilevered beam made of graphite/epoxy and subjected to different types of loading. The beam has a CUS square cross-section with \([+12]_4\) layup. The geometry and mechanical properties are given in Table 3. A comparison of the results with the MSC/NASTRAN finite element analysis of Nixon (1989) is provided in Table 4. The MSC/NASTRAN analysis is based on a 2D plate model. The predictions of the present theory range from +1.7 to -0.7% difference relative to the finite element results.

Table 2. Comparison of flexibility coefficients of NABSA, TAIL and present (lb. in. units)

Table 3. Geometry and mechanical properties of thin-walled beam with \([+12]_4\) CUS square cross-section

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>24.0 in</td>
</tr>
<tr>
<td>Width = depth</td>
<td>1.17 in</td>
</tr>
<tr>
<td>Ply thickness</td>
<td>0.0075 in</td>
</tr>
<tr>
<td>( E_{11} = E_{22} = E_{33} )</td>
<td>11.65 Msi</td>
</tr>
<tr>
<td>( G_{12} = G_{13} )</td>
<td>0.82 Msi</td>
</tr>
<tr>
<td>( G_{23} )</td>
<td>0.7 Msi</td>
</tr>
<tr>
<td>( \nu_{12} = \nu_{13} )</td>
<td>0.05</td>
</tr>
<tr>
<td>( \nu_{23} )</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Table 4. MSC/NASTRAN and present solutions for a CUS cantilevered beam with \([+12]_4\) layups subjected to various tip-load cases

<table>
<thead>
<tr>
<th>Tip load</th>
<th>Tip deformation</th>
<th>% Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial force (100 lb)</td>
<td>Axial disp.</td>
<td>+0.6 %</td>
</tr>
<tr>
<td>Axial force (100 lb)</td>
<td>Twist: 0.3178 deg.</td>
<td>+1.7 %</td>
</tr>
<tr>
<td>Torsional moment (100 lb·in.)</td>
<td>Twist: 2.959 deg.</td>
<td>+1.32 %</td>
</tr>
<tr>
<td>Transverse force (100 lb)</td>
<td>Deflection: 1.866 in.</td>
<td>-0.7 %</td>
</tr>
</tbody>
</table>
Anisotropic thin-walled beams

![Graph](image)

Fig. 4. Bending slope of an anti-symmetric [15]_t cantilever under 1 lb transverse tip load.

For a CUS configuration, the extension–torsional response is decoupled from bending. Since C is constant and g1 does not affect the stiffness coefficients, the flexibility coefficients controlling extension and twist response, S_{11}, S_{12} and S_{22}, coincide with those of Atilgan (1989) and Rehfield and Atilgan (1989). As a consequence, the axial-displacement and twist-angle predictions coincide. However, the lateral deflection under transverse load differs. The tip lateral deflection predicted using the theory of Rehfield (1985), Atilgan (1989) and Rehfield and Atilgan (1989) is 1.724 in., resulting in ~7.6% difference compared to the NASTRAN result.

The test data appearing in the comparisons of Figs 4-9 are reported by Chandra et al. (1990) and Smith and Chopra (1990, 1991). Figures 4 and 5 show the bending

![Graph](image)

Fig. 5. Bending slope of a symmetric [30]_t cantilever under 1 lb transverse tip load.

COE 2:57-1
slope variation along the beam span for antisymmetric and symmetric cantilevers under a 1 lb. transverse tip load. The beam geometry and material properties are given in Table 5. The analytical predictions reported by Smith and Chopra (1990, 1991), and the results obtained on the basis of the analyses of Rehfield (1985), Rehfield and Atilgan (1989), Atilgan (1989) and the present work, are combined in Figs 4 and 5. Results show that the predictions of the present theory are the closest to the test data when compared to the other analytical approaches.

Table 5. Cantilever geometry and properties

| Width | 0.953 in. | $E_{11} = 20.59$ Msi, $E_{22} = E_{33} = 1.42$ Msi |
| Depth | 0.53 in.  | $G_{12} = G_{13} = 0.87$ Msi, $G_{23} = 0.7$ Msi |
| Ply thickness | 0.005 in. | $v_{12} = v_{13} = 0.42, v_{23} = 0.5$ |

Fig. 6. Twist of a symmetric [30]_s cantilever under 1 lb transverse tip load.

Fig. 7. Twist of a symmetric [45]_s cantilever under 1 lb transverse tip load.
Anisotropic thin-walled beams

Fig. 8. Bending slope at mid-span under unit tip torque of symmetric layup cantilever beams.

The bending slope in Figs 4 and 5 is defined in terms of the cross-sectional rotation for theories including shear deformation. For the geometry and material properties considered, this effect is negligible, as shown in Figs 4 and 5 where the span-wise slope at the fixed end predicted by theories with shear deformation is indistinguishable from zero. The nonzero value shown by the test data may be due to the experimental set-up used to achieve clamped-end conditions.

The span-wise twist distribution of a symmetric cantilevered beam with [30]_6 and [45]_6 layups is plotted in Figs 6 and 7, respectively. The beams are subjected to a transverse tip load of 1 lb. Their dimensions and material properties are given in Table 5.
Results show that the present theory and the works of Rehfied and Atilgan (1989) and Atilgan (1989) are the closest to the test data. A similar behavior is found for the bending slope and the twist angle at the mid-span of the symmetric cantilevered beams appearing in Figs 8 and 9. The beams are subjected to a tip torque of 1 lb-in.

CONCLUSION

An anisotropic thin-walled closed-section beam theory has been developed, based on an asymptotical analysis of the shell energy functional. The displacement field is not assumed a priori and emerges as a result of the analysis. In addition to the classical out-of-plane torsional warping, two new contributions are identified, namely axial strain and bending warping. A comparison of the derived governing equations confirms the theory developed by Reissner and Tsai. In addition, explicit closed-form expressions for the beam stiffness coefficients, the stress and displacement fields are provided. The predictions of the present theory have been validated by comparison with finite element simulation, other closed-form analyses, and test data.

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REFERENCES


APPENDIX

In this appendix expressions for some of the relevant variables used in the development as well as the stiffnesses \( C_{ij} (i, j = 1, 4) \) in eqn (42) are provided.

The three stiffness parameters \( A, B \) and \( C \) in eqn (30) are expressed in terms of the Hookean tensor \( E^{ij} \) as follows:

\[
A(s) = \langle D^{1111} \rangle - \frac{(D^{1122})^2}{(D^{2222})} \\
B(s) = 2\langle D^{1122} \rangle - \frac{(D^{1122})(D^{1222})}{(D^{2222})} \\
C(s) = 4\langle D^{1212} \rangle - \frac{(D^{1212})^2}{(D^{2222})}
\]  

(A1)

The 2D Young's moduli \( D^\alpha_{ij} \) are given by

\[
D^\alpha_{ij} = E^\alpha_{ij} - \frac{G^\alpha_{ij} E_{3333}}{E_{3333}} - H_{\alpha\beta} G^\beta_{ij} G^{\beta\alpha}
\]  

(A2)

where

\[
G^\alpha_{ij} = E^\alpha_{ij} - \frac{E_{3333} E_{3333}}{E_{3333}}
\]

and \( H_{\alpha\beta} \) are components of the inverse of the 2D matrix

\[
\begin{bmatrix}
E^{1111} & E^{1133} \\
E^{1333} & E^{3333}
\end{bmatrix}
\]

Combining eqns (34) and (A1), the variables \( b \) and \( c \) can be written as

\[
b(s) = -\frac{(D^{1112}) - (D^{1122})(D^{1222})}{(D^{2222}) - ((D^{1212})^2/(D^{2222}))}
\]

and

\[
c(s) = \frac{1}{4((D^{1112}) - ((D^{1212})^2/(D^{2222})))}
\]  

(A3)

where the angular brackets denote integration over the thickness as defined in eqn (9).

Expressions for the stiffness coefficients \( C_{ij} (i, j = 1, 4) \) in terms of the cross-sectional geometry and materials properties are as follows:

\[
C_{11} = \int \left( A - \frac{B^2}{C} \right) dz \frac{1}{4} \left( B/C \right) \frac{1}{(1/C)} ds
\]

\[
C_{12} = \frac{1}{4} \left( B/C \right) \frac{1}{(1/C)} ds A_e
\]

\[
C_{13} = -\int \left( A - \frac{B^2}{C} \right) \zeta ds \frac{1}{4} \left( B/C \right) \frac{1}{(1/C)} ds
\]

\[
C_{14} = -\int \left( A - \frac{B^2}{C} \right) \gamma ds \frac{1}{4} \left( B/C \right) \frac{1}{(1/C)} ds
\]

\[
C_{22} = \frac{1}{4} \left( B/C \right) \frac{1}{(1/C)} ds A_e
\]

\[
C_{23} = -\frac{1}{4} \left( B/C \right) \frac{1}{(1/C)} ds A_e
\]

\[
C_{24} = -\frac{1}{4} \left( B/C \right) \gamma ds \frac{1}{4} \left( B/C \right) \frac{1}{(1/C)} ds A_e
\]
\[ C_{33} = \int \left( A - \frac{B^2}{C} \right) z^2 \, ds + \int \frac{\left( B/C \right) z \, ds}{\frac{1}{1/C} \, ds} \]

\[ C_{34} = \int \left( A - \frac{B^2}{C} \right) y \, ds + \int \frac{\left( B/C \right) y \, ds}{\frac{1}{1/C} \, ds} \]

\[ C_{44} = \int \left( A - \frac{B^2}{C} \right) y^2 \, ds + \int \frac{\left( B/C \right) y^2 \, ds}{\frac{1}{1/C} \, ds}. \]