On failure of continuum plasticity theories on small scales

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Abstract

We argue that continuum plasticity theories cannot describe plastic behavior if the characteristic length of the problem becomes of the order of the decorrelation length of the dislocation structure.

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It is well recognized that classical plasticity does not work on the microlevel. For example, the hardness predicted by classical plasticity does not depend on the indenter size while experiments show a considerable growth of hardness for the indenter sizes smaller than about 50 μm. In order to apply continuum theory at small scales some extensions of classical plasticity were proposed, in particular, the so-called “strain-gradient plasticity” models. In this note, we present some doubts on the principle possibility for using continuum plasticity theories on such small scales.

Our arguments are of quite a general nature. Let ϕ be a characteristic of micro-inhomogeneous body, for example, dislocation density. In a homogenized description of the body one uses “homogenized” values of ϕ, ̄ϕ. These values are part of the system of equations describing the macro-behavior of the body. Further, any theory of macroscopic behavior, in addition to the formulation of the governing equations, should set up a rule to measure the macro-characteristics experimentally. If experiments allow for the measuring of local fields, ϕ(x), then the experimental values of the “homogenized” field are defined usually as “empirical averages”, i.e. as integrals of the micro-field over some box B of the size l:

\[ \bar{\phi}(x) = \frac{1}{l^3} \int_B \phi(x + \tilde{x}) d\tilde{x} \equiv \langle \phi \rangle_x \]  

For piece-wise functions, the rule (1) corresponds to computing the arithmetic average. We will call Eq. (1) the conventional average. The length l is usually referred to as the size of the representative volume.

Macro-behavior is assumed to be deterministic. So are the functions ̄ϕ(x). On the other hand, ̄ϕ(x) is linked to the random field ϕ(x) by Eq. (1). Consistency of these two features of ̄ϕ(x) puts a constraint on the admissible random field ϕ(x) and the size of the representative volume: the only possible are those for which ̄ϕ(x) practically does not change from one realization to another. This constraint can be expressed in terms of the correlation function of fluctuations ϕ'(x) = ϕ(x) − ̄ϕ(x):

\[ K(x, \tilde{x}) = \mathcal{M} \phi'(x) \phi'(\tilde{x}) \]  

where \( \mathcal{M} \) stands for mathematical expectation. Namely,

\[ \frac{1}{l^6} \int_B \int_B K(x + x_1, x + x_2) d^3x_1 d^3x_2 / A^2 \ll 1 \]  

Here A is a characteristic value of the field ̄ϕ(x).
The condition (3) follows from the identity
\[ \mathcal{M}(\phi(x) - \mathcal{M}\phi(x))^2 = \frac{1}{\mathcal{P}} \int_B \int_B K(x + x_1, x + x_2) d^3x_1 d^3x_2 \]
and means that the variance of \( \phi(x) \) is much smaller than \( \mathcal{M}\phi(x) \).

Condition (3) takes a simpler form for a statistically homogeneous field \( \phi \) when the correlation function depends only on the difference of the arguments \( K(x_1, x_2) = K(x_1 - x_2) \). To satisfy Eq. (3) the correlation function must decay so fast that the values of the random field \( \phi(x) \) are practically uncorrelated on the distances \( a \) which are much smaller than \( l \). Then, the vanishing of the variance of the homogenized field \( \mathcal{M}\phi(x) \) is a form of the law of large numbers. Assuming that \( K(x) \) decays fast enough one can introduce the characteristic length \( a \) as
\[ a^3 = \int K(x) d^3x / A \]
(4)

One can call \( a \) either the correlation length emphasizing that the values of the field are correlated on the distances less than \( a \), or the decorrelation length if the fact of statistical independence of the field over the distances greater than \( a \) is of major importance. The latter term is more appropriate for our purposes. The condition (3) takes the form:
\[ a \ll l \]
(5)

For further discussion we also need a notion of the characteristic length of the macro-problem, \( L \). Roughly, this is the length on which the macro-parameters of the problem change appreciably. For brevity, we give a precise definition and continue the discussion for the one-dimensional case; an extension to three dimensions obviously follows.

Let a smooth function \( \phi(x) \) be defined on a segment \([a, b]\). The quantity
\[ \text{MC}[\phi] = \max_{x,x' \in [a,b]} |\phi(x) - \phi(x')| \]
is called the maximum change of the function \( \phi(x) \) on the segment \([a, b]\). If the number \( \mathcal{L} \) is small enough then the inequality holds:
\[ \frac{d\phi(x)}{dx} \leq \frac{\text{MC}[\phi]}{\mathcal{L}} \]
(6)

Let us then increase \( \mathcal{L} \) up to the maximum possible value that satisfies Eq. (6). This maximum value is called the characteristic length of the field \( \phi(x) \). If one needs to estimate also higher derivatives of \( \phi(x) \), these derivatives must be included in the definition of the characteristic length:
\[ \frac{d\phi(x)}{dx} \leq \frac{\text{MC}[\phi]}{\mathcal{L}}, \quad \frac{d^2\phi(x)}{dx^2} \leq \frac{\text{MC}[\phi]}{\mathcal{L}^2}, \quad \ldots \]
(7)
and the characteristic length \( L \) is the largest possible constant in the system of inequalities (7). Similarly, if a number of fields needs to be evaluated, all of the corresponding inequalities must be included in the system (7).

A key point for our consideration is the inequality:
\[ l \leq L \]
(8)
which holds for all fields homogenized in a conventional way (1). Indeed, consider the derivative of the field \( \phi(x) \),
\[ \frac{d\phi(x)}{dx} = \frac{\phi(x + l/2) - \phi(x - l/2)}{l} \]
We have
\[ \left\langle \frac{d\phi(x)}{dx} \right\rangle = \frac{\phi(x + l/2) - \phi(x - l/2)}{l} \]
(9)

An unwanted feature of the conventional averaging (1) is that it does not remove all fast-oscillating components of the field: this is clearly seen from Eq. (9) where the right-hand side contains the fast-oscillating terms \( \phi(x + l/2) \) and \( \phi(x - l/2) \). To eliminate this oscillation, one does additional averaging of Eq. (9):
\[ \left\langle \frac{d\phi(x)}{dx} \right\rangle = \frac{\phi(x + l/2) - \phi(x - l/2)}{l} \]
(10)

Obviously, from Eq. (10),
\[ \left\langle \frac{d\phi(x)}{dx} \right\rangle \leq \frac{\text{MC}[\phi]}{l} \]
(11)
The characteristic length \( L \) is the largest possible constant in the denominator of the right-hand side of Eq. (11). Therefore, \( L \) is not less than \( l \) as claimed.

Combining the inequalities (5) and (8) we obtain the constraint for the decorrelation length \( a \) relative to the characteristic length of the problem \( L \):
\[ a \ll L \]
(12)

In dislocation plasticity, the fields \( \phi \) are the fields of plastic deformation, elastic deformation and dislocation density tensor. Micrographs (see, e.g., [1]) indicate clearly that the decorrelation length is, typically, in the order of 10–50 \( \mu \)m and may go even higher. This means that, if one takes \( l \) in Eq. (1) in the range 10–50 \( \mu \)m and \( \phi \) as a field of dislocation density, then \( \phi(x) \) will be practically random and cannot be a field of continuum theory which is deterministic by its nature. Therefore, one cannot expect a resolution of such scales in a continuum theory. In hardness tests the dislocation structure is not well documented, though we expect that \( a \) is also in the range of 10–50 \( \mu \)m. If this is so, the application of the strain-gradient plasticity models to such scales is doubtful. Circumstantial evidence for such a conclusion is the scattering of the experimental data for small samples. One may argue that, though strain-gradient plasticity does not make sense on such scales, it may predict the integral characteristics, like...
hardness: for it happens quite often that equations work well beyond the area for which they were derived. A problem with such an argument is that the strain-gradient plasticity models include some new material characteristics, and when applied to small scales, these characteristics hardly make sense. Another possibility for justifying the application of strain-gradient plasticity models to small scales is to reject the conventional rule of measuring the macro-fields and to interpret the macroscopic fields as ensemble averages. Such a view seems possible, but it would require a complete reconsideration of the foundations of the theory.

Reference