An asymptotic theory of sandwich plates

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Abstract

Elastic plates can be described by a two-dimensional theory, if the characteristic length of the stress state along the plate, \( l \), is much bigger than the plate thickness, \( h \). If all elastic moduli of a laminated plate are of the same order, no matter how many lamina the plate has, then the normal to the mid-surface of the plane remain normal in the course of deformation, and the deformation of the plate can be described by the classical plate theory. The situation changes, when the elastic moduli are of different orders of magnitude. This occurs, in particular, for the hard-skin plates, i.e. the sandwich plates the faces of which are very hard. Due to the low deformability of the skin, normal fibers cannot remain normal to the mid-surface in the course of deformation. The deviations are characterized by transverse shear. The difference from the theory of transverse shear, introduced by Timoshenko and Reissner, is that the transverse shear effects are not the corrections to classical plate theory; they are the effects of the leading order. That is caused by the presence of an additional small parameter, the ratio of elastic moduli of the core and the skin. The additional small parameter changes the character of the asymptotics. In this paper, the governing two-dimensional equations for sandwich plates are derived by an asymptotic analysis of linear three-dimensional elasticity. We show that the classical plate theory works only within a certain range of parameters. Beyond that range the asymptotic theory differs from the classical one. We focus especially on the hard-skin plates, but obtain also the universal relations, which can be applied for any values of elastic moduli and the relative thickness of the skin and the core. As an example four-point bending problem is discussed.

Key words: Sandwich plate, asymptotic, membrane

Introduction

This paper came out of an attempt to estimate the slope of the load-displacement curve in four-point bending of sandwich plates. The classical theory \(^1\) gives an

\(^1\) By classical plate theory we mean a two-dimensional theory in which the deformation of plates is described by the displacements of the mid-surface, and only the

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order of magnitude error in this problem. That is in apparent contrast to a
typical error of plate theories of the order \( h/l \), where \( h \) is the plate thickness
and \( l \) the characteristic distance on which the load changes. More careful look
at this problem brings a probable explanation: the classical beam and plate
theory was developed under the assumption that the elastic moduli can change
in the cross-sectional directions, but this change does not bring new small pa-
rameters that are additional to the basic small parameter of the theory,

\[
\frac{h}{l} \ll 1.
\]  

For sandwich plates usually two new parameters come into play, the ratio of
elastic moduli of the core and the skin, \( \mu_c/\mu_s \), which can be of the order of
several hundreds or thousands (by \( \mu \) we denote shear modulus; indices \( c \) and
\( s \) mark values in the core and the skin), and the ratio of the thicknesses of
the skin and the core, \( h_s/h_c \), which can be of the order \( 0.1 - 0.01 \). In vari-
ous industrial applications, the sandwich plates of this type, hard-skin plates,
proved to be especially beneficial. Apparently, the classical plate theory may
not work in such cases, and a question arises what is the leading asym-
ptotic two-dimensional plate theory. At first glance, studying of this question
nowadays may seem unnecessary, because commercial software can be used
to study the three-dimensional stress state of the laminated plates. Alterna-
tively, one can use two-dimensional plate theories with a complicated set of
kinematic parameters [22,7,8,12,16]. However, in either way, the leading effects
that make the most contribution to the stress state remain masked, while the
asymptotic analysis reveals the major mechanisms of the elastic response. Be-
sides, in contrast to static problems, the dynamic three-dimensional problems
still remain computationally quite difficult. Thus, simplifications, caused by
retaining only the leading effects, are quite useful. This paper aims to derive
the governing two-dimensional equations of the hard-skin plates from linear
three-dimensional elasticity theory. We will obtain also "a universal theory",
which can be applied as the leading approximation for any values of elastic
moduli and any ratio \( h_s/h_c \).

We will see that the key role in developing an asymptotic theory is played by
the parameter

\[
\Lambda = \frac{\mu_c h_c}{\mu_s h_s}.
\]

It has the meaning of the ratio of effective extensional rigidities of the core and
the skin. We will call the sandwich plates, for which \( \Lambda \) is small, the hard-skin
plates.

An interplay of the two small parameters, \( \Lambda \) and \( h/l \), turns out to be crucial

leading terms of the equations are retained. The term classical beam theory has a
similar meaning.
in bending problems. It is convenient to introduce a skin stiffness parameter \( \alpha \) by the relation

\[
\frac{\mu_c h_c}{\mu_s h_s} = \left( \frac{h}{L} \right)^\alpha.
\]

(2)

Here \( h_s \) could be of the order of \( h_c \) (thick skin) or much smaller than \( h_c \) (thin skin). For hard-skin plates, \( \alpha \) is a positive number. We will show that in bending problems there are three qualitatively and quantitatively different situations:

\[
0 < \alpha < 2, \quad \alpha = 2, \quad \alpha > 2.
\]

(3)

"Hardness" of the skin increases as \( \alpha \) grows: the bigger \( \alpha \) the bigger \( \mu_s h_s \) relative to \( \mu_c h_c \). In the case of "not very hard skin", \( 0 < \alpha < 2 \), the plate can be described in the leading approximation by the classical plate theory, though various simplifications appear due to the presence of additional small parameters. The case of "very hard skin", \( \alpha > 2 \), is quite different: the plate behaves as a membrane (even in the absence of the extension forces). In both cases, \( \alpha < 2 \) and \( \alpha > 2 \), the transverse shear is determined uniquely in terms of the displacements of the mid-plane. For "not very hard skin" (\( \alpha < 2 \)), transverse shear follows the Kirchhoff hypothesis of classical plate theory: the normal fibers rotate to stay normal to the deformed mid-plane. For "very hard skin" (\( \alpha > 2 \)), the skin prevents the normal fibers to rotate, and the normal fibers remain normal to the initial mid-plane. The intermediate case, \( \alpha = 2 \), is exceptional: the transverse shears cannot be determined explicitly and remain additional kinematic parameters to be found from a boundary value problem. This is shown schematically in Fig.1.

We call the case \( \alpha > 2 \) the membrane regime because the stress state is described in this case by membrane equation.

The sharp change of the sandwich behavior at \( \alpha = 2 \) occurs only in the limit \( h/l \to 0 \). For real plates \( h/l \) is small but finite. Therefore, a smooth transition from "classical" to "non-classical" sandwich behavior is observed. This transition is illustrated in Fig.2 for the case of four-point bending problem. In

\[\text{For symmetric sandwich plates } h = h_c + 2h_s.\]
this Figure $R$ is the ratio of the slopes of load-displacement curves found by classical theory and by asymptotic theory developed in this paper. The Figure shows the dependence of $R$ on the skin stiffness parameter $\alpha$ for three values of $h/l: h/l = 0.1, h/l = 0.01$ and $h/l = 0.001$, and thin skin, $h_s/h_c = 0.05$. The skin stiffness $\alpha$ is defined in terms of sandwich parameters as

$$\alpha = \ln \left( \frac{\mu_c h_c}{\mu_s h_s} \right) / \ln \left( \frac{h}{l} \right).$$

According to Fig.2 for sandwich plates of moderate thickness, $h/l = 0.1$, classical theory predicts the slope of the load-displacement curve with the errors of less than 5% for $\alpha \leq 0.6$. For bigger $\alpha$ the error grows and for $\alpha = 2$ is 55%. For $\alpha = 4$ the slope by classical theory two order of magnitude higher than by asymptotic theory. Note that the typical values of $\alpha$ in four-point bending of carbon fiber/epoxy/Rohacell PMI closed cell foam are equal to about 4-5. For thinner sandwich plates, $h/l = 0.01$, classical theory has the error less than 5% in a larger range of $\alpha : \alpha \leq 1.3$. For very thin sandwich plates, $h/l = 0.001$, the skin must be extremely hard in order for the classical theory to fail. Such sandwich plates are not currently in industrial use.

Emphasize that the skin stiffness $\alpha$ depends not only on the physical characteristics, but also on the geometry of the sandwich plate, and on the external forces (through the characteristic length of the stress state, $l$). Therefore, the same sandwich plate can be described by classical theory for some force, but for another force may need a modified theory. The skin stiffness $\alpha$ decreases as $l$ increases. This is why one can apply classical plate theory for sufficiently large $l$. The membrane regime appears for sufficiently short characteristic length (but still $l \gg h$). The smaller parameter $h/l$, the more pronounced is the differ-
ence between the two regimes and the sharpness of the transition at $\alpha = 2$. We derive a two-dimensional theory, which is valid for any value of $\alpha$ and a universal two-dimensional theory which is the leading approximation for all values of elastic moduli, $\mu_s \sim \mu_c$ and $\mu_c \ll \mu_s$.

The asymptotic analysis shows that in extension problems, the leading approximation is always given by classical plate theory. The deviations from classical plate theory become noticeable only in higher order terms. The leading correction is related to the core thickening. To characterize these corrections, we introduce parameter $\beta$ by the relation

$$\frac{\mu_c \eta_{c}}{\mu_s \eta_{s}} = \left(\frac{\sqrt{h_s h_c}}{l}\right)^{\beta},$$

and show that the core thickening is determined explicitly for $\beta < 4$ and for $\beta > 4$, while for $\beta = 4$ one has to solve a boundary value problem to find the core thickening.

We obtain also similar results for anisotropic plates, when the material has a plane of material symmetry that is parallel to the mid-plane of the plate in the undeformed state, and within each lamina the elastic moduli are of the same order. For simplicity, we focus on symmetric sandwich plates.

We consider here only static problems. However, our results are easily extended to low-frequency vibrations, i.e. the vibrations, the characteristic frequency of which, $\omega$, satisfies the condition

$$\omega \sim \frac{1}{l} \sqrt{\mu_c / \rho_c},$$

where $\rho_c$ is mass density of the core. In the low frequency case, the two-dimensional dynamical equations are obtained from the static equations by adding the leading inertia terms to the equations of statics [5]. If (5) does not hold, then a two-dimensional theory must include high-frequency effects [3,6,14,19,20,21].

In the next two Sections we formulate the results for all the cases mentioned; in Section 3 we solve an auxiliary problem on the deformation of elastic layer with the prescribed displacements at the faces; that allows us to formulate a leading two-dimensional approximation for an elastic laminated plate with arbitrary geometry and arbitrary physical parameters (Section 4). This approximation employs only one small parameter, $h/l$. The approximation of Section 4 is analyzed further using the presence of additional small parameters in Section 5 (bending of isotropic plates), Section 6 (bending of anisotropic plates), and Section 7 (extension of isotropic and anisotropic plates). In Section 8 we consider a possible extension of the theory to sandwich beams (a complete study of this problem requires a special analysis), and in Section 9
we solve the four-point bending problem within the framework of asymptotic universal theory and discuss some outcomes. The equations are derived by the variational-asymptotic method which was suggested in [4] and further applied for various problems in [14,11,25,26,27,24,5].

1 Governing equations for isotropic plates

Denote by $x^1, x^2$ the Cartesian coordinates in the mid-plane, and by $x$ the normal coordinate to the mid-plane. We use index notation and write also $x^\alpha$ for the in-plane coordinates (3). The Greek indices run though values 1, 2. The projections of the displacement vector of the mid-plane on the mid-plane and the normal are denoted by $u^\alpha(x^\beta)$ and $u(x^\beta)$, respectively. The displacements of a point of the plate with coordinates $x^\alpha, x$ are denoted by $w^\alpha(x^\alpha, x)$, $w(x^\alpha, x)$. Summation over repeated upper and low indices is implied.

In the linear elasticity theory, the solution of any three-dimensional problem for a plate can be presented as a superposition of two solutions, one with even function of $x, w(x^\alpha, x)$, and odd functions of $x, w^\alpha(x^\beta, x)$, and another one with odd function of $x, w(x^\alpha, x)$, and even functions of $x, w^\alpha(x^\beta, x)$. The first one will be called a bending problem, the second one an extension problem. The corresponding split of the load is shown symbolically in Fig. 4.

3 It is assumed that elastic properties are symmetric with respect to the planes that are parallel to the mid-plane, and elastic moduli are even functions of $x$, i.e. the sandwich is symmetric.
As was mentioned, the leading approximation in the extension problem is given by the classical plate theory. This is shown in Section 7. Therefore, here we summarize the results only for the bending problems.

Consider first the isotropic plates. The elastic properties are characterized by the following dependence of Lame’s constants, $\lambda$ and $\mu$, on the normal coordinate.

\[
\mu(x) = \begin{cases} 
\mu_c & |x| < h_c/2 \\
\mu_s & h_c/2 < |x| < h_c/2+h_s
\end{cases}, \quad \lambda(x) = \begin{cases} 
\lambda_c & |x| < h_c/2 \\
\lambda_s & h_c/2 < |x| < h_c/2+h_s
\end{cases}.
\]

In case of thin skin ($h_s \ll h_c$), in all asymptotic estimates, $h_c$ can be replaced by $h = h_c + 2h_s$.

The plate is loaded at the top face by a surface force with components $P, P^\alpha$. The external tangent force at the faces, $P^\alpha$, is assumed to be on the order of the normal force, $P$, or smaller:

\[
P^\alpha = O(P). \tag{6}
\]

The edge conditions are not essential in the derivation of the leading approximation.

First, let the skin be thin, i.e. one neglect corrections on the order of $h_s/h_c$ near unity. As was mentioned, there are three different cases (3).

**Case $\alpha < 2$.** In this case the displacement distribution over the normal coordinate is

\[
w = u - \begin{cases} 
\frac{1}{2}\sigma_c \left( \frac{h_s^2}{4} - x^2 \right) \Delta u & |x| \leq h_c/2 \\
\frac{1}{2}\sigma_s \left( \frac{h_s^2}{4} - x'^2 \right) \Delta u & |x'| \leq h_s/2
\end{cases}, \quad w_\alpha = -u_\alpha x, \quad |x| \leq h/2. \tag{7}
\]

Here $x'$ is a local normal coordinate in the skin,

\[x' = x - \frac{1}{2} (h_c + h_s),\]

$\sigma$ a coefficient linked to Poisson’s coefficient,

\[\sigma = \frac{\nu}{1-\nu},\]

$\Delta$ Laplace’s operator, comma before a Greek index denotes partial derivative with respect to in-plane coordinates:

\[u_{\alpha} \equiv \frac{\partial u}{\partial x^\alpha}.\]
The displacement field has the only field variable, \( u(t,x^\alpha) \). It satisfies the Kirchhoff equation,

\[
\mu_s h_s h_c^2 (1 + \sigma_s) \Delta^2 u = P.
\]

This is the Euler equation of the functional

\[
\int_{\Omega} \left[ \frac{\mu_s h_s h_c^2}{2} \left( \sigma_s \Delta u^2 + u_{\alpha\beta} u^{\alpha\beta} \right) - P u \right] d\Omega.
\]

Specifying the geometrical constraints at the boundary and varying the functional (9) one gets the boundary conditions as well.

Note that the bending rigidity in the functional (9), \( \mu_s h_s h_c^2 \), coincides with the usual expression,

\[
\int_{-h/2}^{h/2} \mu x^2 dx = \mu_c \frac{h_c^3}{12} + \frac{2}{3} \mu_s \left( \left( \frac{h_c}{2} + h_s \right)^3 - \left( \frac{h_c}{2} \right)^3 \right),
\]

when \( \mu_c h_c \ll \mu_s h_s \) and \( h_s \ll h_c \).

Emphasize that the quadratic terms in (7) (and in further formula (20)) must be taken into account despite their being much smaller than the previous leading term: these small quadratic terms in the displacement distribution give a finite contribution to energy.

For low-frequency vibrations, the only difference in the governing equation is the additional leading inertia term,

\[
\mu_s h_s h_c^2 (1 + \sigma_s) \Delta^2 u + \frac{\rho \partial^2 u}{\partial t^2} = P,
\]

where \( \rho \) is the mass density per unit area. The transition from statics to low-frequency dynamics in all other cases considered further is similar, and, thus, will not be mentioned again.

Case \( \alpha > 2 \). In this case, in the core

\[
w = u, \quad w_\alpha = - \left( \langle u,_{\alpha} \rangle + \hat{\omega} e_{\alpha\beta} x^\beta \right) x.
\]

Here \( \langle \cdot \rangle \) is the averaging over the plate

\[
\langle \cdot \rangle = \frac{1}{|\Omega|} \int_{\Omega} \cdot d\Omega, \quad |\Omega| = \int_{\Omega} d\Omega,
\]

\( e_{\alpha\beta} \) Levi-Civita symbol \((e_{11} = e_{22} = 0, e_{12} = -e_{21} = 1)\), \( \hat{\omega} \) a constant \(^4\),

\[
\hat{\omega} = - \langle u,_{\alpha} x^\beta \rangle e^{\alpha\beta} / I, \quad I = \langle x,_{\alpha} x^{\alpha} \rangle.
\]

\(^4\) To simplify the formulas, the coordinates \( x^\alpha \) are chosen in such a way that \( \langle x^{\alpha} \rangle = 0 \).
So, the in-plane displacements, $w_\alpha$, are linear functions of the normal coordinate, $x$. The coefficients of these linear functions depend linearly on the in-plane coordinates $x^\alpha$. The functions contain three constants, $\langle u, \alpha \rangle$ and $\dot{\omega}$. It is assumed in (10) that the boundary conditions of the three-dimensional elasticity problem at the plate edge do not yield the kinematic constraints on the displacements of the core. Otherwise, the constants $\langle u, \alpha \rangle$ and $\dot{\omega}$ are zero, and

$$w_\alpha = 0.$$  \hspace{1cm} (13)

Equation (13) holds also, if the core displacements are not constrained at the edge, but the plate is clamped at the edge in normal direction (i.e. $u = 0$ at the edge). Then the constants, $\langle u, \alpha \rangle$ and $\dot{\omega}$, are equal to zero due to the divergence theorem.

For zero constants, $\langle u, \alpha \rangle$ and $\dot{\omega}$, the normal to the mid-plane in the undeformed state maintains its direction in space. If the plate is clamped only at a part of the boundary (e.g., as a cantilever), then $\langle u, \alpha \rangle$ and $\dot{\omega}$ can be non-zero.

In the upper skin, $x > 0$, the displacements are:

$$w = u, \quad w_\alpha = -\left(\langle u, \alpha \rangle + \dot{\omega}e_{\alpha\beta}x^\beta\right)\frac{h_c}{2} - x'u_\alpha.$$ \hspace{1cm} (14)

There are additional terms that provide continuity of the displacements (10) and (14) at the interface, but they give small contribution to energy.

The governing equation for $u(x^\alpha)$ is the membrane equation

$$\mu_c h_c \Delta u - \frac{\mu_s h_s^3}{12}(1 + \sigma_s) \Delta^2 u + P = 0.$$ \hspace{1cm} (15)

Note the difference between the bending rigidities in (15) and (8). The first two terms of (15) have the orders,

$$\mu_c h_c \bar{u} \quad \text{and} \quad \frac{\mu_s h_s^3 \bar{u}}{10l^4},$$

where $\bar{u}$ is the order of displacements. The ratio of the second term to the first one is on the order of $\mu_s h_s^3/10\mu_c h_cl^2$. If the skin is so thin that

$$\left(\frac{h_s}{l}\right)^2 \ll 10\frac{\mu_c h_c}{\mu_s h_s},$$ \hspace{1cm} (16)

then the second term in (15) can be dropped and (15) transforms to the membrane equation,

$$\mu_c h_c \Delta u + P = 0.$$ \hspace{1cm} (17)

In many cases the condition (16) holds. For example, for $h_s/h_c = 1/10$, $h_c/l = 1/10$, the estimate (16) becomes $10^{-6} \ll \mu_c/\mu_s$, which is usually true. However,
since our task is to cover all cases when the transition from three-dimensional to two-dimensional theory is possible, there is no reason to ignore the second term in (15). For a localized load, this term describes the boundary layer. The width of the boundary layer \( l^* = \sqrt{\mu_s h^3_s/10 \mu_c h_c} \), must be much larger than \( h \) in order to believe the effects described by the second term of (15). Therefore,

\[
\frac{\mu_s}{10 \mu_c} \left( \frac{h_s}{h} \right)^3 \gg 1. \tag{18}
\]

In principle, there are materials for which (16), (18) hold, but they do not seem to be in industrial use.

Equation (15) is the Euler equation of the functional,

\[
\int_\Omega \left[ \frac{\mu_s h^3_s}{6} \left( \sigma_s \Delta u^2 + u_{,\alpha \beta} u^{,\alpha \beta} \right) + \frac{\mu_c h_c}{2} \right. \\
\left. \left( u_{,\alpha} - \langle u_{,\alpha} \rangle - \tilde{\omega} e_{\alpha \beta} x^\beta \right) \left( u^{,\alpha} - \langle u^{,\alpha} \rangle - \tilde{\omega} e^{,\alpha \gamma} x_\gamma \right) - Pu \right] \, d\Omega.
\]

Note that the constants, \( \langle u_{,\alpha} \rangle \) and \( \tilde{\omega} \), though not appearing in the Euler equation, enter the boundary conditions, which are obtained by varying the functional (19).

The first term in (19) is the bending energy of the skin, the second one is the shear energy of the core. The variation of the bending energy brings the second term of (15). As was mentioned, it is usually negligible. Accordingly, the energy functional simplifies to

\[
\int_\Omega \left[ \frac{\mu_c h_c}{2} \right. \left. \left( u_{,\alpha} - \langle u_{,\alpha} \rangle - \tilde{\omega} e_{\alpha \beta} x^\beta \right) \left( u^{,\alpha} - \langle u^{,\alpha} \rangle - \tilde{\omega} e^{,\alpha \gamma} x_\gamma \right) - Pu \right] \, d\Omega.
\]

Case \( \alpha = 2 \). The displacement distribution over the thickness is

\[
w = u + \begin{cases} 
\frac{1}{2} \sigma_c \left( \frac{h^2}{4} - x^2 \right) \psi^{,\alpha}_{,\alpha} & |x| \leq h_c/2 \\
-\frac{1}{2} \sigma_s \left( \frac{h^2}{4} - x^2 \right) \Delta u & |x'| \leq h_s/2,
\end{cases}
\tag{20}
\]

\[
w_{,\alpha} = \begin{cases} 
x \psi^{,\alpha}_{,\alpha} & |x| \leq h_c/2 \\
+\frac{1}{2} h_c \psi^{,\alpha}_{,\alpha} + \frac{1}{2} \sigma_s \left( \frac{h^2}{4} - x^2 \right) \frac{h_s}{2} \psi^{,\alpha}_{,\alpha} & |x'| \leq h_s/2, \quad x > 0 \\
-\frac{1}{2} h_c \psi^{,\alpha}_{,\alpha} - \frac{1}{2} \sigma_s \left( \frac{h^2}{4} - x^2 \right) \frac{h_s}{2} \psi^{,\alpha}_{,\alpha} & |x'| \leq h_s/2, \quad x < 0
\end{cases}
\]

Kinematics of the plate is described by two functions, \( u(x^{,\alpha}) \) and \( \psi^{,\alpha}_{,\alpha}(x^{,\beta}) \). In contrast to the case \( \alpha < 2 \), where \( \psi^{,\alpha}_{,\alpha} = -u_{,\alpha} \), and to the case \( \alpha > 2 \), where
\( \psi_\alpha \) is a linear function of in-plane coordinates, for \( \alpha = 2 \) and \( \psi_\alpha \) are linked by a system of differential equations,

\[
\frac{1}{2} \mu_s h_s h_c^2 \left[ (1 + 2\sigma_s) \left( \psi_{,\beta} \right)_\alpha + \Delta \psi_\alpha \right] + \mu_c h_c (\psi_\alpha + u_\alpha) = 0 \tag{21}
\]

\[
\mu_c h_c (\psi_\alpha + u_\alpha)_\alpha + P = 0. \tag{22}
\]

The boundary conditions can be obtained by varying the functional\(^5\)

\[
\int_\Omega \left[ \frac{\mu_s h_s h_c^2}{2} \left( \sigma_s (\psi_\alpha^2) + \psi_{(\alpha,\beta)} \psi_{(\alpha,\beta)} \right) + \frac{\mu_c h_c}{2} (\psi_\alpha + u_\alpha) (\psi_\alpha + u^\alpha) - Pu \right] d\Omega.
\]

**Universal approximation for hard-skin plates with the thin skin.** Any asymptotic theory assumes some limit procedure. For example, the equations of the classical plate theory may be thought of as equations for displacements of the mid-plane, which are obtained in the limit \( h \to 0 \). In this limit procedure, one should fix the dependence of forces on the space coordinates and reduce the magnitude of forces in such a way that that the displacements remain finite. In the case of hard-skin plates, one should simultaneously change elastic moduli and the ratio \( h_s/h_c \) to keep the shear parameter \( \alpha \) constant. Though such asymptotic procedure helps to recognize the major elastic effects involved, in practical problems all the characteristics of the plate are given and are not a subject of a limit procedure. Therefore, it is desirable to have a two-dimensional theory, which is applicable for any \( \alpha \). We call such theory universal. It is easy to see that all three cases are the approximations of a two-dimensional theory with the energy functional

\[
I = \int_\Omega \left[ \frac{\mu_s h_s^3}{6} \sigma_s \Delta u^2 + u_{\alpha\beta} u^{\alpha\beta} \right] d\Omega - \frac{\mu_s h_s h_c^2}{2} \sigma_s (\psi_\alpha^2) + \psi_{(\alpha,\beta)} \psi_{(\alpha,\beta)} + \frac{\mu_c h_c}{2} (\psi_\alpha + u_\alpha) (\psi_\alpha + u^\alpha) - Pu \right] d\Omega. \tag{23}
\]

For example, to obtain the relations of the case \( \alpha > 2 \), we note that for \( \alpha > 2 \) the leading term of the functional (23), which contains \( \psi_\alpha \), is

\[
\int_\Omega \frac{\mu_s h_s h_c^2}{2} \sigma_s \left( \psi_\alpha^2 + \psi_{(\alpha,\beta)} \psi_{(\alpha,\beta)} \right) d\Omega.
\]

Its minimum is zero. It is achieved at the function, \( \psi_\alpha \), that are solutions of

\[\psi_{(\alpha,\beta)} = \frac{1}{2} \left( \frac{\partial \psi_\alpha}{\partial x^\beta} \right) + \frac{\partial \psi_\alpha}{\partial x^\alpha}.\]
the equations
\[
\frac{\partial \psi_\alpha}{\partial x^\beta} + \frac{\partial \psi_\beta}{\partial x^\alpha} = 0. \tag{24}
\]
The general solution of (24) is
\[
\psi_\alpha = \hat{\psi}_\alpha - \hat{\omega} e_{\alpha\beta} x^\beta, \tag{25}
\]
where \(\hat{\psi}_\alpha, \hat{\omega}\) are some constants. Considering functional (23) on the fields (25) we obtain the functional,
\[
\int_\Omega \left[ \frac{\mu_s h^3}{6} \left( \sigma_s \Delta u^2 + u_{\alpha\beta} u^{\alpha\beta} \right) \\
+ \frac{\mu_c h_c}{2} \left( u_{\alpha} + \hat{\psi}_\alpha - \hat{\omega} e_{\alpha\beta} x^\beta \right) \left( u^\alpha + \hat{\psi}^\alpha - \hat{\omega} e^{\alpha\beta} x_\gamma \right) - Pu \right] d\Omega. \tag{26}
\]
The functional (26) must be minimized over the constants \(\hat{\psi}_\alpha\) and \(\hat{\omega}\). The corresponding equations for \(\hat{\psi}_\alpha\) and \(\hat{\omega}\) are
\[
\int_\Omega \left( u_{\alpha} + \hat{\psi}_\alpha - \hat{\omega} e_{\alpha\beta} x^\beta \right) d\Omega = 0 \tag{27}
\]
\[
\int_\Omega \left( u_{\alpha} + \hat{\psi}_\alpha - \hat{\omega} e_{\alpha\beta} x^\beta \right) e^{\alpha\beta} x_\gamma d\Omega = 0.
\]
Hence, \(\hat{\psi}_\alpha = -\langle u_{\alpha} \rangle, \hat{\omega}\) is given by (12), and (26) transforms to (19).

A universal theory for all possible values of elastic moduli and lamina thicknesses is formulated in the next Section.

**Historic remarks.** The universal theory is a Reissner’s type theory, which involves the transverse shear. However, in contrast to the original transverse shear theory, where the incorporation of the transverse shear allows one to take into account the corrections on the order \((h/l)^2\), the proposed theory is the theory of the leading approximation. In transverse shear theory the arguments were given in favor of different values for the shear factor, like \(5/6 \mu h\) or \(\pi^2/12\mu h\). Numerically, that does not bring considerable difference in the results. In the theory of the leading approximation a proper choice of every coefficient is essential. In particular, the shear factor in the functional (23) is equal to
\[
k = \mu_c h_c. \tag{28}
\]
The major mechanical effects described by functional (23) are well recognized in the engineering literature, as one can see from a remarkably clear treatment of the subject given by H.G. Allen [2]. However, a universal theory, which is valid in every case when a two-dimensional description is possible, does not
seem to have been developed. Usually, the authors address a specific class of problems neglecting the effects that are not important for this class, but could become pronounced for other problems. Perhaps, the most close to the energy functional (23) is the energy functional proposed by Libove and Batdorf [15]. The theory presented by Allen [2] and the theory given in the Handbook of sandwich construction [1] coincides with the universal asymptotic theory if the vector field \( \psi_\alpha \) in (23) is potential and the bending energy of the skin is dropped. Theories presented in some modern textbooks differ, however, more significantly. This is caused by the treatment of the transverse shear as a correction to classical plate theory and not as a leading effect. Consider, for example, how this is done by Vinson [22], p.52. In theories taking into account the transverse shear, the shear force \( q_\alpha \) is assumed to be a linear function of the transverse shear \( \psi_\alpha + u_\alpha \):

\[
q_\alpha = k(\psi_\alpha + u_\alpha).
\]

The shear factor \( k \) is assumed to be an average value of the layer shear moduli with some weight, \( f(x) \),

\[
k = \int_{-h/2}^{h/2} f(x) \mu(x) dx.
\] (29)

To compute the shear factor, in [22] the the following weight function was used,

\[
f(x) = \frac{5}{4} \left( 1 - \left( \frac{x}{h/2} \right)^2 \right).
\]

For a sandwich plate that gives

\[
k = \mu_c \frac{5}{4} h_c \left( 1 - \frac{h_c^2}{3h^2} \right) + \frac{5}{2} \mu_s \frac{h_s^2}{h} \left( 1 + O \left( \frac{h_s}{h} \right) \right).
\]

If \( h_s \ll h_c \),

\[
k = \frac{5}{6} \mu_c h_c + \frac{5}{2} \mu_s \frac{h_s}{h} = \frac{5}{6} \mu_c h_c \left( 1 + 3 \frac{\mu_s h_s}{\mu_c h_c} \right).
\] (30)

The second term in (30) is much smaller than the first one if \( \mu_s \sim \mu_c \). Then (30) yields Reissner’s result \( k = 5/6 \mu_c h_c \). For hard-skin plates, the second term in (30) can be much bigger than the first one, and (30) even approximately is not equal to (28). Note that no smooth weight function yields (28): since \( \mu_s h_s \gg \mu_c h_c \), the major contribution to the integral (29) is given by the skin area while according to the asymptotic formula (28) the shear factor is determined by the characteristics of the core. Similar finite deviations of the coefficients of two-dimensional equations from the asymptotic values can be found in other textbooks (see, e.g., [9],[18]).
2 A universal theory of sandwich plates

The equations of the previous Section hold for hard-skin plates with a thin skin. In practice, there are plates with intermediate values of parameters, when the skin is hard but "not very hard" or thin, but not "very thin". Therefore, it is desirable to have a two-dimensional theory which works for all values of parameters: \( \mu_s \) could be on the order of \( \mu_c, h_s \) on the order of \( h_c \); but the cases \( \mu_c \ll \mu_s \) and \( h_s \ll h_c \) should also be included. The energy functional of such theory, derived in Section 5, is

\[
I = \int_\Omega \left[ \frac{\mu_s h_s^3}{6} \left( \sigma_s \Delta u^2 + u_{,\alpha \beta} u^{\alpha \beta} \right) + \frac{\mu_s h_s}{2} \left( \sigma_s \left( h_c \psi_{,\alpha}^\alpha - h_s \Delta u \right)^2 + \left( h_c \psi_{(, \alpha} \psi_{\beta)} - h_s u_{,\alpha \beta} \right) \left( h_c \psi_{(, \alpha} \psi_{\beta)} - h_s u_{,\alpha \beta} \right) \right] \left( \sigma_c \left( \psi_{,\alpha}^\alpha \right)^2 + \psi_{,\alpha \beta} \psi_{\alpha \beta} \right) + \frac{\mu_c h_c}{12} \left( \psi_{,\alpha} + u_{,\alpha} \right) \left( \psi_{,\alpha} + u_{,\alpha} \right) - Pu \right] d\Omega.
\]  (31)

In special cases considered it transforms to the functionals of the previous Section. In the case of hard-skin thick plates the fourth term can be omitted. In the case considered by classical theory of plates, \( \mu_s \sim \mu_c \), the functional can be simplified using the presence of the small parameter, \( h/l \). The leading term of the functional is shear energy,

\[
\int_\Omega \frac{\mu_c h_c}{2} (\psi_{,\alpha} + u_{,\alpha}) (\psi_{,\alpha} + u_{,\alpha}) d\Omega.
\]

Minimum of the shear energy is zero and achieved for

\[
\psi_{,\alpha} = -u_{,\alpha}.
\]  (32)

The functional (31) on the field (32) is

\[
\int_\Omega \left[ A (\Delta u)^2 + B u_{,\alpha \beta} u^{\alpha \beta} - Pu \right] d\Omega  \]  (33)

where

\[
A = \frac{\mu_s h_s^3}{6} \sigma_s + \frac{\mu_s h_s (h_c + h_s)^2}{2} \sigma_s + \frac{\mu_c h_c^3}{12} \sigma_c = \int_{-h/2}^{h/2} \mu(x) \sigma(x) x^2 dx
\]  (34)

\[
B = \frac{\mu_s h_s^3}{6} + \frac{\mu_s h_s (h_c + h_s)^2}{2} + \frac{\mu_c h_c^3}{12} = \int_{-h/2}^{h/2} \mu(x) x^2 dx.
\]

The functional (33) with the coefficients (34) is the energy functional of the classical plate theory.
The particular value of the shear factor, \( c^2_{\text{c}} \), in (31) is not essential for the theory to be correct in the case \( c_{\text{s}} \sim c^2_{\text{c}} \). However, this value is important to embed the case of the hard-skin plates.

The energy functional of a universal theory of anisotropic sandwich plates is given in Section 6.

### 3 Asymptotic analysis of an elastic layer with the prescribed displacements of the faces

We begin the derivation with a solution of an auxiliary static problem: find the leading terms of the asymptotics for a homogeneous elastic layer, occupying the region \( x^\alpha \in \Omega, \ -h/2 \leq x \leq h/2 \), when the displacements are prescribed at the faces \( x = \pm h/2 \):

\[
   w_{\alpha}\big|_{x=\pm h/2} = w_{\alpha}^\pm \left( x^\beta \right) \quad w\big|_{x=\pm h/2} = w^\pm \left( x^\beta \right).
\]

We seek the asymptotics away from the plate edges. The boundary conditions at the edges do not affect this asymptotics. According to the Saint-Venant principle for the prescribed displacements at the boundary, any edge load exponentially decays away from the edge at the distances on the order of \( h \), and does not influence the interior stress state. The characteristic length \( \ell \), of functions \( w_{\alpha}^\pm, w^\pm \) is assumed to be much larger than \( h \).

For material which possesses a plane of elastic symmetry parallel to the midplane, the problem splits in a sum of the bending problem and the extension problem. This is symbolically shown in Fig.4.

\[
   w = w^+, \ w_\alpha = w_\alpha^+ \quad w = w^+, w^-/2, w_\alpha = w_\alpha^+ - w_\alpha^-/2 \quad w = w^+, w^-/2, w_\alpha = w_\alpha^+ + w_\alpha^-/2
\]

Fig. 5. Split of a problem with the prescribed face displacements into the sum of the bending problem and the extension problem.

In the bending problem, \( w \) and \( w_{\alpha} \) are even and odd functions of \( x \), respectively; in the extension problem, \( w \) is an odd function of \( x \), while \( w_{\alpha} \) are even functions.

The true displacements minimize the free energy functional

\[
   \int_{\Omega} \int_{-h/2}^{h/2} F \left( \varepsilon_{\alpha\beta}, \varepsilon_{\alpha 3}, \varepsilon_{33} \right) \, d\Omega \, dx,
\]

See the definition of the characteristic length in [5] (Section 14.4).
\[ \varepsilon_{\alpha\beta} \equiv \frac{1}{2} \left( \frac{\partial w_\alpha}{\partial x^\beta} + \frac{\partial w_\beta}{\partial x^\alpha} \right), \quad \varepsilon_{\alpha3} \equiv \frac{1}{2} \left( \frac{\partial w_\alpha}{\partial x} + \frac{\partial w}{\partial x^\alpha} \right), \quad \varepsilon_{33} \equiv \frac{\partial w}{\partial x}, \quad (37) \]
onumber

on the set of displacements extracted by the constraints (35).

Let first the material be isotropic. For an isotropic material the free energy density is:

\[ F = \frac{1}{2} \lambda (\varepsilon_\alpha^2 + \varepsilon_{33}^2) + \mu \left( \varepsilon_{\alpha\beta} \varepsilon^{\alpha\beta} + 2\varepsilon_{\alpha3} \varepsilon_{3\alpha3} + \varepsilon_{33}^2 \right). \quad (38) \]

Following the general scheme [4,5], we split \( F \) into the sum of two functions \( F_u \) and \( F_\perp \):

\[ F = F_u + F_\perp \quad (39) \]

\[ F_u (\varepsilon_{\alpha\beta}) \equiv \min_{\varepsilon_{\alpha\beta}} F (\varepsilon_{\alpha\beta}, \varepsilon_{\alpha3}, \varepsilon_{33}) = \mu \left( \sigma (\varepsilon_\alpha^2 + \varepsilon_{\alpha3} \varepsilon_{3\alpha3}) \right), \]

\[ F_\perp \equiv F - F_u = 2\mu \varepsilon_{\alpha3} \varepsilon_{3\alpha3} + \frac{1}{2} (\lambda + 2\mu) (\varepsilon_{33} + \sigma \varepsilon_\alpha^2). \]

Consider first the bending problem. In the bending problem the displacements satisfy the constraints at the faces:

\[ w \big|_{x=\pm h/2} = u, \quad w_\alpha \big|_{x=\pm h/2} = \pm \psi_\alpha \frac{h}{2}. \quad (40) \]

Here we introduced the notation

\[ u \equiv \frac{w^+ + w^-}{2}, \quad \psi_\alpha \equiv \frac{w^+ - w^-}{h}. \quad (41) \]

Denote the order of \( u \) by \( \bar{u} \). We assume that \( \psi_\alpha \) remain finite as \( h \to 0 \). Since \( \psi_\alpha \) are dimensionless we set \( \psi_\alpha \sim \bar{u}/l \). We are going to find energy which includes terms of the order \( \mu h (\bar{u}/l)^2 \) and the corrections of the order \( \mu h (\bar{u}/l)^2 (h/l)^2 \).

To formulate the result it is convenient to introduce the transverse shear, \( \varphi_\alpha \), as

\[ \varphi_\alpha = \psi_\alpha + \frac{\partial u}{\partial x^\alpha}, \]

and the modified lateral displacement, \( u^* \), and the transverse shear, \( \varphi^*_\alpha \),

\[ u^* = u + \frac{h^2}{12} \left[ \sigma \frac{\partial \psi_\alpha}{\partial x^\alpha} + \frac{1 - \sigma}{2} \frac{\partial \varphi_\alpha}{\partial x^\alpha} \right], \quad \varphi^*_\alpha = \psi_\alpha + \frac{\partial u^*}{\partial x^\alpha}. \]

Then energy is given by the formula,

\[ \Phi = \int_{-h/2}^{h/2} F \, dx = \frac{\mu h^3}{12} \left( \sigma \left( \psi_\alpha^2 + \psi_{(\alpha,\beta)} \psi^{(\alpha,\beta)} \right) + \frac{\mu h}{2} \varphi^*_\alpha \varphi^*_{\alpha} \right) \]

\[ + \frac{(\lambda + 2\mu) (1 - \sigma) h^3}{96} \left( \frac{\partial \varphi_{\alpha}}{\partial x^\alpha} \right)^2. \quad (42) \]
Proceeding to the derivation of (42), we denote the orders of \( w \) and \( w_\alpha \) by \( \tilde{w} \) and \( \tilde{w}_\alpha \), respectively. We assume that \( \tilde{w} \sim \tilde{u} \) and \( \tilde{w}_\alpha \sim h\tilde{u}/l \). The consistency of such an assumption will be seen from the asymptotic analysis of the problem. The strains have the orders:

\[
\varepsilon_{\alpha\beta} \sim \frac{h\tilde{u}}{l^2}, \quad \varepsilon_{\alpha\beta} \sim \frac{\tilde{u}}{l}, \quad \varepsilon_{33} \sim \frac{\tilde{u}}{h}.
\]

Therefore, the leading term of energy is

\[
\frac{\lambda + 2\mu}{2} \left( \frac{\partial w}{\partial x} \right)^2.
\]

Minimization of energy,

\[
\int_\Omega \int_{-h/2}^{h/2} \frac{\lambda + 2\mu}{2} \left( \frac{\partial w}{\partial x} \right)^2 \, d\Omega dx,
\]

on the set of all displacements \( w \) which satisfy the boundary conditions (40) yields

\[ w = u. \]

Then we seek the next approximation as

\[ w = u + w', \]

where \( w' \ll u \). All the leading terms are in \( F_\perp \),

\[
F_\perp = \frac{\mu}{2} \left( \frac{\partial w_\alpha}{\partial x} + \frac{\partial u}{\partial x^\alpha} + \frac{\partial w'}{\partial x^\alpha} \right) \left( \frac{\partial w_\alpha}{\partial x} + \frac{\partial u}{\partial x^\alpha} + \frac{\partial w'}{\partial x^\alpha} \right) \frac{\lambda + 2\mu}{2} \left( \frac{\partial w'}{\partial x} + \sigma x \frac{\partial w'}{\partial x^\alpha} \right)^2.
\]

Since \( w' \ll u \), to determine \( w_\alpha \) we have to minimize the functional

\[
\int_\Omega \int_{-h/2}^{h/2} \frac{\mu}{2} \left( \frac{\partial w_\alpha}{\partial x} + \frac{\partial u}{\partial x^\alpha} + \frac{\partial w'}{\partial x^\alpha} \right) \left( \frac{\partial w_\alpha}{\partial x} + \frac{\partial u}{\partial x^\alpha} + \frac{\partial w'}{\partial x^\alpha} \right) \, d\Omega dx,
\]

on the set of all displacements \( w_\alpha \), which satisfy the boundary conditions (40). The minimizer is

\[ w_\alpha = \psi_{\alpha \cdot} x. \]

(44)

Plugging this in (43) we get the functional to be minimized to find \( w' \) :

\[
\int_\Omega \int_{-h/2}^{h/2} \left[ \frac{\mu}{2} \left( \varphi_{\alpha} + \frac{\partial w'}{\partial x^\alpha} \right) \left( \varphi_{\alpha} + \frac{\partial w'}{\partial x^\alpha} \right) + \frac{\lambda + 2\mu}{2} \left( \frac{\partial w'}{\partial x} + \sigma x \frac{\partial w'}{\partial x^\alpha} \right)^2 \right] dx d\Omega.
\]

(45)

Admissible functions \( w' \) are equal to zero on the faces. The interaction term,

\[
\mu \varphi_{\alpha} \frac{\partial w'}{\partial x^\alpha},
\]
cannot be neglected in determination of \( w' \) because it is of the same order as the interaction term
\[
(\lambda + 2\mu) \sigma x \frac{\partial w'}{\partial x} \frac{\partial \psi^\alpha}{\partial x^\alpha}.
\]
Integrating by parts we put the interaction term in the form,
\[
- \mu w' \frac{\partial \varphi^\alpha}{\partial x^\alpha}.
\]
So, to find \( w' \) we have to minimize the functional
\[
\int_\Omega \int_{-h/2}^{h/2} \left[ \frac{\lambda + 2\mu}{2} \left( \frac{\partial w'}{\partial x} \sigma x \frac{\partial \psi^\alpha}{\partial x^\alpha} \right)^2 - \mu w' \frac{\partial \varphi^\alpha}{\partial x^\alpha} \right] dxd\Omega.
\]
The minimizer is the solution of the differential equation,
\[
\frac{\partial}{\partial x} (\lambda + 2\mu) \frac{\partial w'}{\partial x} + (\lambda + 2\mu) \sigma \frac{\partial \psi^\alpha}{\partial x^\alpha} + \mu \frac{\partial \varphi^\alpha}{\partial x^\alpha} = 0,
\]
with zero conditions at \( x = \pm h/2 \). We get
\[
w' = \frac{1}{2} \left( \frac{h^2}{4} - x^2 \right) \left[ \frac{\partial \psi^\alpha}{\partial x^\alpha} + \frac{1 - \sigma}{2} \frac{\partial \varphi^\alpha}{\partial x^\alpha} \right]. \tag{46}
\]
Here we used that \( \mu/(\lambda + 2\mu) = (1 - \sigma)/2 \). Then the first terms of the asymptotic expansion of \( w \) are
\[
w = u + \frac{1}{2} \left( \frac{h^2}{4} - x^2 \right) \left[ \frac{\partial \psi^\alpha}{\partial x^\alpha} + \frac{1 - \sigma}{2} \frac{\partial \varphi^\alpha}{\partial x^\alpha} \right]. \tag{47}
\]
We rewrite (47) to make zero the average over the thickness of the second term
\[
w = u^* + \frac{1}{2} \left( \frac{h^2}{12} - x^2 \right) \left[ \frac{\partial \psi^\alpha}{\partial x^\alpha} + \frac{1 - \sigma}{2} \frac{\partial \varphi^\alpha}{\partial x^\alpha} \right]. \tag{48}
\]
This is how the modified lateral displacement \( u^* \) comes into play.

The substitution of \( w_\alpha \) and \( w \) in (45) yields
\[
\int_\Omega \int_{-h/2}^{h/2} \left[ \frac{\mu}{2} \left( \varphi^\alpha + \frac{1}{2} \left( \frac{h^2}{12} - x^2 \right) \frac{\partial}{\partial x^\alpha} \left( \sigma \frac{\partial \psi^\beta}{\partial x^\beta} + \frac{1 - \sigma}{2} \frac{\partial \varphi^\beta}{\partial x^\beta} \right) \right)^2 \right] dxd\Omega
\]
\[
+ \int_\Omega \frac{\lambda + 2\mu}{2} \frac{h^3}{12} \left( \frac{1 - \sigma}{2} \frac{\partial \varphi^\alpha}{\partial x^\alpha} \right)^2 d\Omega. \tag{49}
\]
Using that the average value of \( h^2/12 - x^2 \) is zero, neglecting the terms of the order \( \mu h^5 (\bar{u}/l)^3 \), and replacing \( \varphi_\alpha \) in the last term of (49) by \( \varphi^*_\alpha \) (this is possible within the accuracy accepted), we obtain the last two terms of (42).
At the next step of the variational-asymptotic method we seek the minimizer in the form
\[ w = w_1 + w'' , \quad w_\alpha = \psi_\alpha x + w_\alpha' \] (50)
where \( w_1 \) is given by (48) and \( w'' \ll w_1, w_\alpha' \ll \psi_\alpha x \). We plug (50) in (39) and keep only the leading terms, containing \( w'' \) and \( w_\alpha' \), and the leading interaction terms between \( w', w_\alpha' \) and \( u, \psi_\alpha \). The leading terms of \( w_\alpha' \) in \( F_\parallel \)
are much smaller than the leading terms of \( w_\alpha' \) in \( F_\perp \),
\[ \frac{1}{2} \frac{\partial w_\alpha'}{\partial x} \frac{\partial w_\alpha'}{\partial x} . \]
Let the orders of \( w'' \) and \( w_\alpha' \) be \( \bar{w}'' \) and \( \bar{w}_\alpha' \), respectively. The interaction terms between \( w_\alpha' \) and \( \psi_\alpha \) in \( F_\parallel \)
have the order \( \bar{w}_\alpha' \mu h (\bar{u}/l^3) \). The interaction term between \( w_\alpha' \) and \( \varphi_\alpha^* \) in \( F_\perp \) vanish because \( w_\alpha' \) are zero on the faces. Other interaction terms are of the order \( \bar{w}_\alpha' \mu h (\bar{u}/l^3) \). Let us neglect the interaction term between \( w'' \) and \( w_\alpha' \). Then the order of \( w_\alpha' \) is obtained by equating the orders of the interaction terms, \( \bar{w}_\alpha' \mu h (\bar{u}/l^2) \), and the order of the leading term, \( \mu (\bar{w}_\alpha')^2 / h^2 \). We obtain \( \bar{w}_\alpha' \sim h^3 \bar{u}/l^3 \), i.e. \( w_\alpha' \) are \( (h/l)^2 \) times smaller than the first term in (50). Similarly, \( \bar{w}'' \) is \( (h/l)^2 \) times smaller than the first term in (50). Taking into account the interaction terms between \( w'' \) and \( w_\alpha' \) does not change the orders of \( w'' \) and \( w_\alpha' \). So, the energy contributions of \( w'' \) and \( w_\alpha' \) are beyond the accuracy considered. Finally, the displacements are given by (44) and (48) while energy is determined by (42). The first term of (42) is the integral of \( F_{ii} \) over \( x \) computed on the fields (44).

Formula (42) includes all the terms of the orders \( \mu h (\bar{u}/l)^2 \) and \( \mu h (h \bar{u} l^2)^2 \). Emphasize that such a simple formula holds only if the modified lateral displacement, \( u^* \), and modified transverse shear, \( \varphi^* \), are used. In what follows, we need only the leading approximation of energy. In the leading approximation,
\[ \Phi = \frac{\mu h^3}{12} \left( \sigma (\psi_\alpha')^2 + \psi_{(\alpha,\beta)} \psi^{(\alpha,\beta)} \right) + \frac{\mu h}{2} \varphi_\alpha \varphi^\alpha \] (51)
Indeed, the last term in (42) is always small in comparison with the previous one and can be dropped. To simplify the first two terms of (42) we note that \( \psi_\alpha \) and \( \varphi_\alpha \) are independent fields. If \( \psi_\alpha \) and \( \varphi_\alpha \) are of the same order then the
The first term in (42) can be dropped while $u^*$ and $\varphi_\alpha^*$ can be replaced by $u$ and $\varphi_\alpha$. Then

$$\Phi = \frac{\mu h}{2} \varphi_\alpha \varphi^\alpha.$$  

The first term of (42) can be of the same order as the second one if $\varphi_\alpha$ are small. In this case

$$\varphi_\alpha^* \sim \frac{h}{l} \psi_\alpha.$$

Hence,

$$\psi_\alpha = -\frac{\partial u}{\partial x_\alpha} + O \left( \frac{h \bar{u}}{l^2} \right), \quad u^* = u + O \left( \frac{h \bar{u}}{l^2} \right),$$

and $u^*, \varphi_\alpha^*$ can be replaced by $u$ and $\varphi_\alpha$, respectively. We arrive at (51).

Similarly, for the extension problem one obtains in the leading approximation

$$w = \psi x, \quad w_\alpha = u_\alpha + \frac{1}{2} \left( \frac{h^2}{4} - x^2 \right) \frac{\partial \psi}{\partial x_\alpha}, \quad u_\alpha \equiv \frac{w^+_\alpha + w^-_\alpha}{2}, \quad \psi \equiv \frac{w^+ - w^-}{h},$$

$$\Phi = \int_{-h/2}^{h/2} F dx = \mu h \left( \sigma \left( u_\alpha^2 + u_{\alpha,\beta} u^{(\alpha,\beta)} \right) + \frac{\lambda + 2\mu}{2} h \left( \psi + \sigma u_\alpha^2 \right)^2 \right).$$

Formulas (51) and (53) determine how energy of an elastic layer depends on the displacements, prescribed at the faces, in the leading approximation.

For anisotropic materials, energy of a layer can be obtained in the same way. For materials possessing a plane of elastic symmetry that is parallel to the faces, free energy density has the form

$$F = \frac{1}{2} E^{\alpha \beta \gamma \delta} \varepsilon_{\alpha \beta} \varepsilon_{\gamma \delta} + \frac{1}{2} E \varepsilon_{33}^2 + 2 G^{\alpha \beta} \varepsilon_{\alpha 3} \varepsilon_{33} + E^{\alpha \beta} \varepsilon_{\alpha \beta} \varepsilon_{33}. \quad (54)$$

The calculation of $F_\parallel$ and $F_\perp$ yields the relations [5]:

$$F = F_\parallel + F_\perp,$$

$$F_\parallel = \min_{\varepsilon_{\alpha,\beta}} F = \frac{1}{2} E^{\alpha \beta \gamma \delta} \varepsilon_{\alpha \beta} \varepsilon_{\gamma \delta}, \quad E^{\alpha \beta \gamma \delta}_{\parallel} = E^{\alpha \beta \gamma \delta} - \frac{1}{E} E^{\alpha \beta} E^{\gamma \delta}, \quad (55)$$

$$F_\perp = F - F_\parallel = 2 G^{\alpha \beta} \varepsilon_{\alpha 3} \varepsilon_{33} + \frac{1}{2} E \left( \varepsilon_{33} + \frac{1}{E} E^{\alpha \beta} \varepsilon_{\alpha \beta} \right)^2.$$  

In the bending problems, in the leading approximation

$$w = u + \frac{1}{2E} E^{\alpha \beta} \psi_{\alpha,\beta} \left( \frac{h^2}{4} - x^2 \right), \quad w_\alpha = \psi_{\alpha, \beta} x$$

$$\Phi = \int_{-h/2}^{h/2} F dx = \frac{h^3}{24} E^{\alpha \beta \gamma \delta}_{\parallel} \psi_{\alpha,\beta} \psi_{\gamma,\delta} + \frac{h}{2} G^{\alpha \beta} \left( \psi_{\alpha} + u_{\alpha} \right) \left( \psi_{\beta} + u_{\beta} \right).$$
In the extension problem, in the leading approximation

\[ w = \psi x \quad w_\alpha = u_\alpha + \frac{1}{2} \left( \frac{h^2}{4} - x^2 \right) \frac{\partial \psi}{\partial x^\alpha} \]  

(57)

\[ \Phi = \int_{-h/2}^{h/2} F dx = \frac{h}{2} E_{\alpha \beta \gamma \delta} u_{\alpha, \beta} u_{\gamma, \delta} + \frac{1}{2} E \left( \psi + \frac{1}{E} E_{\alpha \beta} u_{\alpha, \beta} \right)^2. \]

4 A theory of laminated plates

The relations obtained in the previous Section allow us easily construct the governing two-dimensional equations of laminated plates, which contain all leading effects, but may include smaller effects as well. Denote by \( F_k \left( w^+_{(k)}, w^-_{(k)\alpha} \right) \) the elastic energy of the \( k \)th layer with the prescribed displacements on the upper face, \( w^+_{(k)}, w^+_{(k)\alpha} \), and the lower face, \( w^-_{(k)}, w^-_{(k)\alpha} \). For isotropic materials, according to (51) and (53),

\[ F_k \left( w^+_{(k)}, w^-_{(k)\alpha} \right) = \int_\Omega \left[ \frac{\mu_k h_k^3}{12} \left( \sigma_k \left( \psi^+_{(k)\alpha} \right)^2 + \psi_{(k)\alpha, \beta} \psi_{(k)\beta} \right) + \frac{\mu_k h_k}{2} \left( \psi_{(k)\alpha} + u_{(k)\alpha} \right) \left( \psi_{(k)\alpha} + u^\alpha_{(k)} \right) \right] d\Omega \]

(58)

\[ + \int_\Omega \left[ \mu_k h_k \left( \sigma_k \left( u^\alpha_{(k)\alpha} \right)^2 + u_{(k)\alpha, \beta} u^\beta_{(k)} \right) + \lambda_k + 2\mu_k h_k \left( \psi_{(k)} + \sigma_k u^\alpha_{(k)\alpha} \right)^2 \right] d\Omega, \]

where \( \sigma_k = \lambda_k / ( \lambda_k + 2\mu_k ) \),

\[ u_{(k)} = \frac{w^+_{(k)} + w^-_{(k)}}{2}, \quad u_{(k)\alpha} = \frac{w^+_{(k)\alpha} + w^-_{(k)\alpha}}{2}, \quad \psi_{(k)} = \frac{w^+_{(k)} - w^-_{(k)}}{h_k}, \quad \psi_{(k)\alpha} = \frac{w^+_{(k)\alpha} - w^-_{(k)\alpha}}{h_k}, \]

(59)

\( h_k \) the thickness of the \( k \)th layer, and \( \lambda_k, \mu_k \) its Lame’s constants.

Then, the total energy of the laminated plate, containing \( N \) layers, is

\[ \mathcal{F} \left( w^+_{(1)}, \ldots, w^+_{(N)}, w^+_{(1)\alpha}, \ldots, w^+_{(N)\alpha} \right) = \sum_{k=1}^N \mathcal{F}_k \left( w^+_{(k)}, w^+_{(k)\alpha} \right). \]

(60)

To find a deformed state of the plate subject to the external forces acting on the top face, one has to minimize over \( w^+_{(1)}, \ldots, w^+_{(N)}, w^+_{(1)\alpha}, \ldots, w^+_{(N)\alpha} \) the functional\(^7\)

\[ I = \mathcal{F} \left( w^+_{(1)}, \ldots, w^+_{(N)}, w^+_{(1)\alpha}, \ldots, w^+_{(N)\alpha} \right) - \int_\Omega \left( P w^+_{(1)} + P^\alpha w^+_{(1)\alpha} \right) d\Omega. \]

(61)

\(^7\) We number the layers starting from the top layer.
The functions \( w^\pm_{(1)}, \ldots, w^\pm_{(N)} \) must satisfy the conditions of continuity of displacements on the interfaces:

\[
\begin{align*}
  w^-_{(k)} &= w^+_{(k+1)}, & w^-_{(k)\alpha} &= w^+_{(k+1)\alpha}, & k = 1, \ldots
\end{align*}
\]  

(62)

If the material is anisotropic, then one has to replace (58) with the sum of bending and extension energies (56) and (57).

The energy functional contains small parameters, \( h_k \), and can be simplified further. The result, however, depends on the relative orders of elastic moduli in different layers. We consider in the next Section only the case of symmetric hard-skin sandwich plates. The functional (61) can be a starting point of the asymptotic analysis in all other cases as well.

## 5 Bending of isotropic hard-skin plates

In bending problems, \( w \) is an even function of \( x \), while \( w_\alpha \) are odd functions. Accordingly, the core is subject to bending only, while the skin experiences both bending and extension. Therefore, the number of arguments of the energy functional is reduced, and, as such, one can take the kinematic characteristics of the upper skin, \( u^s, \psi^s_\alpha, u^s_\alpha, \psi^s \), and the kinematic characteristics of the bending problem for the core, \( u^c, \psi^c_\alpha \). In what follows we drop index of the core and write

\[
u \equiv u^c, \quad \psi_\alpha \equiv \psi^c_\alpha,
\]

while for the skin kinematic parameters we use capital letters:

\[
u^s = U, \quad \psi^s_\alpha = \Psi_\alpha, \quad u^s_\alpha = U_\alpha, \quad \psi^s = \Psi.
\]

Since for the skin, according to (41) and (52),

\[
\begin{align*}
  U &= \frac{w^+ + w^-}{2}, & \Psi &= \frac{w^+ - w^-}{h_s}, \\
  U_\alpha &= \frac{w^+_\alpha + w^-_\alpha}{2}, & \Psi_\alpha &= \frac{w^+_\alpha - w^-_\alpha}{h_s},
\end{align*}
\]

the skin surface displacements are expressed in terms of \( U, \Psi, \Psi_\alpha \) as

\[
\begin{align*}
  w^+ &= U + \frac{h_s}{2} \Psi, & w^- &= U - \frac{h_s}{2} \Psi, & w^+_\alpha &= U_\alpha + \frac{h_s}{2} \Psi_\alpha, & w^-_\alpha &= U_\alpha - \frac{h_s}{2} \Psi_\alpha.
\end{align*}
\]

For the core, according to (22),

\[
\begin{align*}
  w^+ &= u, & w^+_\alpha &= \frac{h_c}{2} \psi_\alpha.
\end{align*}
\]
The continuity of displacements put constraints on the admissible functions:

\[ U - \frac{h_s}{2} \psi = u, \quad U_\alpha - \frac{h_s}{2} \psi_\alpha = \frac{h_c}{2} \psi_\alpha. \]  

(64)

The energy functional to be analyzed is

\[
I = 2 \int_\Omega \left[ \frac{\mu_s h_s^3}{12} \left( \sigma_s \left( \Psi_\alpha^\alpha \right)^2 + \Psi_{(\alpha,\beta)} \Psi^{(\alpha,\beta)} \right) + \frac{\mu_s h_s}{2} \left( \Psi_\alpha + U_\alpha \right) \left( \Psi^\alpha + U^\alpha \right) \right] d\Omega 
+ \mu_s h_s \left( \sigma_s \left( U_\alpha^\alpha \right)^2 + U_{(\alpha,\beta)} U^{(\alpha,\beta)} \right) + \left( \lambda_s + 2 \mu_s \right) \frac{h_s}{2} \left( \psi + \sigma_s U_\alpha^\alpha \right)^2 \right] d\Omega 
+ \int_\Omega \left[ \frac{\mu_c h_c^2}{12} \left( \sigma_c \left( \psi_\alpha^\alpha \right)^2 + \psi_{(\alpha,\beta)} \psi^{(\alpha,\beta)} \right) + \frac{\mu_c h_c}{2} \left( \psi_\alpha + u_\alpha \right) \left( \psi^\alpha + u^\alpha \right) \right] d\Omega 
- \int_\Omega \left( P \left( U + \frac{h_s}{2} \Psi \right) + P^\alpha \left( U_\alpha + \frac{h_s}{2} \Psi_\alpha \right) \right) d\Omega.
\]

(65)

The functional depends on nine functions, \( u, \psi_\alpha, U, \Psi_\alpha, U_\alpha, \Psi, \) which are linked by three constraints (64). We can choose six functions \( u, \psi_\alpha, \Psi, \Psi_\alpha \) as independent kinematic variables. Then from (64)

\[ U = u + \frac{h_s}{2} \Psi, \quad U_\alpha = \frac{1}{2} (h_c \psi_\alpha + h_s \Psi_\alpha). \]  

(66)

The work of external force becomes

\[
\int_\Omega \left[ P \left( u + h_s \Psi \right) + P^\alpha \left( \frac{1}{2} h_c \psi_\alpha + h_s \Psi_\alpha \right) \right] d\Omega.
\]

The functional does not contain derivatives of \( \Psi, \) and minimization over \( \Psi \) is an algebraic problem. To shorten the way to the final result, we fix \( u, \psi_\alpha \) and seek \( \Psi, \Psi_\alpha \) minimizing the functional. Suppose that \( \Psi \) can be neglected in the first equation (66), i.e.

\[ U = u, \]  

(67)

and that

\[ U_\alpha \sim hu_\alpha, \quad \Psi_\alpha \sim u_\alpha. \]  

(68)

Besides, let us put \( P^\alpha = 0. \) After determining \( \Psi, \Psi_\alpha, \) from this simplified problem we will check that the resulting equations are consistent with our assumptions. Indeed, minimization over \( \Psi \) yields

\[ \ddot{\Psi} = -\sigma_s U_\alpha^\alpha. \]  

(69)

In minimization over \( \Psi_\alpha, \) the terms with derivatives of \( \Psi_\alpha \) can be dropped as small in comparison with the terms in \( \left( \Psi_\alpha + u_\alpha \right) \left( \Psi^\alpha + u^\alpha \right). \) Hence, the minimizing functions, \( \ddot{\Psi}_\alpha, \) are

\[ \ddot{\Psi}_\alpha = -u_\alpha. \]  

(70)
Plugging (67) - (70) in the energy functional we obtain

\[
\frac{1}{\mu_s h_s} I = 2 \int_{\Omega} \left[ \frac{h_c^2}{12} \left( \sigma_s \Delta u^2 + u_{\alpha\beta} u^{\alpha\beta} \right) + \frac{1}{4} \left( \sigma_s (h_c \psi_\alpha^\alpha - h_s \Delta u)^2 + (h_c \psi_{(\alpha,\beta)} - h_s u_{\alpha\beta}) (h_c \psi^{(\alpha,\beta)} - h_s u^{\alpha\beta}) \right) \right] d\Omega \\
+ \mu_s h_c \int_{\Omega} \left[ \frac{h_c^2}{12} \left( \sigma_c (\psi_\alpha^\alpha)^2 + \psi_{(\alpha,\beta)} \psi^{(\alpha,\beta)} \right) + \frac{1}{2} (\psi_\alpha + u_{\alpha}) (\psi^\alpha + u^{\alpha}) \right] d\Omega \\
- \frac{1}{\mu_s h_s} \int_{\Omega} P u d\Omega.
\]

(71)

The first integral is the energy of the skin, the second one the energy of the core, and the last one the work of external force (all are referred to \( \mu_s h_s \)). In the skin energy, the first term is the bending energy of the skin, and the second one the extension energy. Let us minimize the functional (71) over \( \psi_\alpha \). If \( \alpha < 2 \) then the leading term containing \( \psi_\alpha \) is the core shear energy, \((\psi_\alpha + u_{\alpha}) (\psi^\alpha + u^{\alpha})\). Minimum is achieved at

\[
\psi_\alpha = -u_{\alpha}.
\]

(72)

In the extension energy of a thin skin, \( h_c \psi_{(\alpha,\beta)} - h_s u_{\alpha\beta} \) can be replaced by \(-h_c u_{\alpha\beta} \). The bending energy of the skin and the bending energy of the core can be neglected compared with extension energy of the skin. Finally, we arrive at the functional of classical plate theory, in which small terms are dropped:

\[
I = \int_{\Omega} \left[ \frac{\mu_s h_s h_c^2}{2} \left( \sigma_s \Delta u^2 + u_{\alpha\beta} u^{\alpha\beta} \right) - P u \right] d\Omega.
\]

The energy of the plate in this case is just the extension energy of the skin. If \( \alpha > 2 \), then the leading terms of (71) containing \( \psi_\alpha \) are in the extension energy of the skin

\[
\frac{1}{\mu_s h_s} I = 2 \int_{\Omega} \left[ \frac{1}{4} \left( \sigma_s (h_c \psi_\alpha^\alpha - h_s \Delta u)^2 + (h_c \psi_{(\alpha,\beta)} - h_s u_{\alpha\beta}) (h_c \psi^{(\alpha,\beta)} - h_s u^{\alpha\beta}) \right) \right] d\Omega.
\]

(73)

The minimizer is

\[
\psi_\alpha = \frac{h_s}{h_c} u_{\alpha} + \tilde{\psi}_\alpha,
\]

(74)

where \( \tilde{\psi}_\alpha \) is any solution of the equations

\[
\tilde{\psi}_{\alpha,\beta} + \tilde{\psi}_{\beta,\alpha} = 0.
\]

(75)

The general solution of (75) is

\[
\tilde{\psi}_\alpha = \tilde{\psi}_\alpha + \tilde{\omega}_{\alpha\beta} x^\beta,
\]

(76)
It contains three arbitrary constants, $\psi_\alpha$ and $\omega$. The minimum of the functional (73) is zero.

Functions $\psi_\alpha$ enter in shear energy in the sum with $u_{\alpha}$. Therefore, for thin skin plates, the first term in (74) can be dropped, and the leading terms of energy functional becomes

$$I = \int_\Omega \left[ \frac{\mu c h^3}{6} \left( \sigma_s \Delta u^2 + u_{\alpha \beta} u^{\alpha \beta} \right) + \frac{\mu_c h_c}{2} \left( u_{\alpha} + \psi_\alpha + \omega e_{\alpha \gamma} x^\gamma \right) \left( u^{\alpha} + \psi^\alpha + \omega e^{\alpha \gamma} x_\gamma \right) - P u \right] d\Omega.$$  \hspace{1cm} (77)

The energy consists of the shear energy of the core and the bending energy of the skin. Functional (77) should be minimized over the normal displacements, $u$, and the constants, $\psi_\alpha$ and $\omega$. These constants are determined from equations (27).

For zero $\psi_\alpha$, $\omega$, energy of the plate is identical to membrane energy with the extensional force $\mu_c h_c$. If, additionally, the skin is so thin that

$$\frac{\mu_s h^3_s}{\mu_c h_c t^2} \ll 1,$$  \hspace{1cm} (78)

then the functional simplifies to

$$I = \int_\Omega \left[ \frac{\mu_c h_c}{2} u_{\alpha} u^{\alpha} - P u \right] d\Omega.$$

For a thick skin plates, the term

$$\frac{\mu_c h_c}{\mu_s h_s} \int_\Omega \frac{h^2_s}{12} \left( \sigma_c \left( \psi_\alpha \right)^2 + \psi_{(\alpha, \beta)} \psi^{(\alpha, \beta)} \right) d\Omega$$

considered on the field (74), becomes

$$\frac{\mu_c h_c}{\mu_s h_s} \int_\Omega \frac{h^2_s}{12} \left( \sigma_c \Delta u^2 + u_{\alpha \beta} u^{\alpha \beta} \right) d\Omega.$$

For any positive $\alpha$ it is much smaller than the first term of (71). Finally, for a thick skin plate,

$$I = \int_\Omega \left[ \frac{\mu_s h^3_s}{6} \left( \sigma_s \Delta u^2 + u_{\alpha \beta} u^{\alpha \beta} \right) + \frac{\mu_c h_c}{2} \left( 1 + \frac{h_s}{h_c} \right) u_{\alpha} + \psi_\alpha + \omega e_{\alpha \gamma} x^\gamma \left( 1 + \frac{h_s}{h_c} \right) u^{\beta} + \psi^\beta + \omega e^{\beta \gamma} x_\gamma \right] - P u \right] d\Omega.$$  \hspace{1cm} (79)
The values of the constants, \( \hat{\psi}_\alpha, \hat{\omega} \), change accordingly:

\[
\hat{\psi}_\alpha = -\left(1 + \frac{h_s}{h_c}\right) u_\alpha, \quad \hat{\omega} = -\left(1 + \frac{h_c}{h_s}\right) \langle u_\alpha x_\beta \rangle e^{\alpha\beta}/I.
\]

If \( \alpha = 2 \), then the terms containing \( \psi_\alpha \) in the core shear energy and the skin extension energy are of the same order. To find \( \psi_\alpha \) one has to solve a boundary value problem. Clearly, \( \psi_\alpha \) is going to be on the order of \( u_\alpha \). Thus, the bending energy of the core and the skin can be dropped, the difference, \( h_c \psi_{(\alpha,\beta)} - h_s u_{\alpha\beta} \), in the extension energy of the skin can be replaced for a thin skin by \( h_c \psi_{(\alpha,\beta)} \), and the energy functional for thin skin plates simplifies to

\[
I = \int_\Omega \left[ \frac{\mu_s h_s h_c^2}{2} \left( \sigma_s \left( \psi^{(\alpha)}_{\alpha} \right)^2 + \psi_{\alpha,\beta} \psi_{\alpha,\beta} \right) \right. \\
\left. + \frac{\mu_c h_c}{2} \left( \psi_\alpha + u_\alpha \right) \left( \psi_\alpha + u_\alpha \right) - Pu \right] d\Omega.
\]

To combine all three cases for thin skin plate, one keeps in the functional (71) the skin bending energy, the core shear energy, drop the core bending energy and neglect in the skin extension energy \( h_s u_{\alpha\beta} \). So, we get the functional (23). For a thick skin plate one has to use the functional (79).

To check the validity of the simplifications made we note that the relations (68) follow from the second formula (66) and from (70). Thus, from (69), \( \Psi \sim h \Delta u \), and the neglecting of \( \Psi \) in the first relation (64) was possible indeed. Besides, the terms in the work of external forces containing \( P^\alpha \) are small in comparison with \( P \), if the assumption (6) holds. This justifies the simplifications made.

### 6 Bending of anisotropic hard-skin plates

For anisotropic plates a variety of possible asymptotics becomes much richer. We consider here only the cases, when the elastic moduli within each layer are of the same order. The derivation repeats that of the previous Section and yields the following energy functionals for thin skin plates:

For \( \alpha < 2 \)

\[
\int_\Omega \left[ \frac{h_s h_c^2}{2} E_{\alpha\beta\gamma\delta} u_{\alpha\beta} u_{\gamma\delta} - Pu \right] d\Omega \quad (80)
\]

for \( \alpha > 2 \)

\[
\int_\Omega \left[ \frac{h_c^3}{6} E_{\alpha\beta\gamma\delta} u_{\alpha\beta} u_{\gamma\delta} + \frac{h_c}{2} G_{\alpha\beta} u_{\alpha} u_{\beta} - Pu \right] d\Omega \quad (81)
\]
for $\alpha = 2$

$$
\int_{\Omega} \left[ \frac{h_c h_s}{2} E_{i\beta}^{\alpha \beta \delta} \psi_{\alpha, \beta} \psi_{\gamma, \delta} + \frac{h_c}{2} \frac{h_s}{2} G_c^{\alpha \beta} (\psi_{\alpha} + u_{\alpha}) (\psi_{\beta} + u_{\beta}) - Pu \right] d\Omega \quad (82)
$$

Combining the results, we get for any $\alpha$ the energy functional

$$
\int_{\Omega} \left[ \frac{h_c^3}{6} E_{i\beta}^{\alpha \beta \gamma \delta} u_{\alpha, \beta} u_{\gamma, \delta} + \frac{h_c h_s^2}{2} E_{i\beta}^{\alpha \beta \delta} \psi_{\alpha, \beta} \psi_{\gamma, \delta} + \frac{h_c}{2} G_c^{\alpha \beta} (\psi_{\alpha} + u_{\alpha}) (\psi_{\beta} + u_{\beta}) - Pu \right] d\Omega. \quad (83)
$$

For thick skin plate, the only essential simplification from the general functional of Section 3 is the possibility to find explicitly $\Psi$ and $\Psi_\alpha$:

$$
\Psi = - \frac{1}{E} E_{i\beta}^{\alpha \beta} U_{\alpha}^\alpha, \quad \Psi_\alpha = - u_{\alpha}. \quad (84)
$$

These relations were used in the derivation of (80)-(83). So, the energy functional of anisotropic thick skin plate is

$$
\int_{\Omega} \left[ \frac{h_s^3}{6} E_{i\beta}^{\alpha \beta \gamma \delta} u_{\alpha, \beta} u_{\gamma, \delta} + \frac{h_c h_s^2}{2} E_{i\beta}^{\alpha \beta \delta} \psi_{\alpha, \beta} \psi_{\gamma, \delta} + \frac{h_c}{2} G_c^{\alpha \beta} (\psi_{\alpha} + u_{\alpha}) (\psi_{\beta} + u_{\beta}) - Pu \right] d\Omega.
$$

7 Extension of the isotropic hard-skin sandwich plates

In the extension problem, $w_\alpha$ are even functions of $x$, $w$ an odd function of $x$. Therefore, the core is in extension while the skin can be both bent and stretched. We characterize the kinematics of the sandwich plates by the core parameters, $u_\alpha$ and $\psi$, and the skin parameters, $U_\alpha, \Psi, U, \Psi_\alpha$. Energy is the sum of energy of the core and energy of the skin:

$$
\mathcal{F} = \int_{\Omega} \left[ \mu_c h_c \left( \sigma_c (u_{\alpha, \alpha})^2 + u_{(\alpha, \beta)} u_{(\alpha, \beta)} \right) + \frac{\lambda_c + 2\mu_c h_c}{2} \left( \psi + c u_{\alpha, \alpha}^2 \right) \right] d\Omega \\
+ 2 \int_{\Omega} \left[ \frac{\mu_c h_s^3}{12} \left( \sigma_s (\psi_{\alpha, \alpha})^2 + \psi_{\alpha, \beta} \psi_{\alpha, \beta} \right) + \frac{\mu_s h_s}{2} (\psi_\alpha + U_\alpha) (\psi_\beta + U_\beta) \right. \\
+ \mu_s h_s \left. \left( \sigma_s (U_{\alpha, \alpha})^2 + U_{(\alpha, \beta)} U_{(\alpha, \beta)} \right) + \frac{\lambda_s + 2\mu_s h_s}{2} \left( \psi + \sigma_s U_{\alpha, \alpha}^2 \right) \right] d\Omega. \quad (85)
$$

\text{8} We use for anisotropic case the same notation for kinematic parameters as in Section 5.
The continuity of the displacements on the interfaces allows us to express $U_{\alpha}$ and $U$ in terms of other parameters:

$$U_{\alpha} = u_{\alpha} + \frac{h_s}{2} \psi_{\alpha}, \quad U = \frac{1}{2} (h_c \psi + h_s \Psi). \quad (86)$$

Functions $u_{\alpha}, \psi, \Psi$ may be considered independent. Let us assume that the terms $h_s \psi_{\alpha}$ and $h_s \Psi$ in (86) are small, thus setting

$$U_{\alpha} = u_{\alpha}, \quad U = \frac{1}{2} h_c \psi. \quad (87)$$

We put also $P = 0$. Keeping in the energy functional only the leading terms in $\Psi$ and $\psi_{\alpha}$ and leading interaction terms we obtain

$$\Psi_{\alpha} = -U_{\alpha} = -\frac{h_c}{2} \psi_{\alpha}, \quad \Psi = -\sigma_s U_{\alpha}^\alpha = -\sigma_s u_{\alpha}. \quad (88)$$

The energy functional becomes,

$$\frac{1}{\mu_s h_s} I = \frac{\mu_c h_c}{\mu_s h_s} \int_{\Omega} \left[ \sigma_c \left( \frac{u_{\alpha}^\alpha}{2} + u_{(\alpha,\beta)} u^{(\alpha,\beta)} \right) + \frac{\lambda_c + 2\mu_c}{2\mu_c} \left( \psi + \sigma_c u_{\alpha}^\alpha \right)^2 \right] d\Omega$$

$$+ 2 \int_{\Omega} \left[ \frac{h_s^2}{12} \frac{h_c^2}{4} \left( \sigma_s \Delta \psi^2 + \psi_{\alpha \beta} (\psi^{\alpha \beta}) + \sigma_s \left( \frac{u_{\alpha}^\alpha}{2} + u_{(\alpha,\beta)} u^{(\alpha,\beta)} \right) \right] d\Omega$$

$$- \frac{1}{\mu_s h_s} \int_{\Omega} P^\alpha u_{\alpha} d\Omega. \quad (89)$$

If $\beta < 4$, then the leading term in $\psi$ is the core "thickening" energy, $(\psi + \sigma_c u_{\alpha}^\alpha)^2$. Minimization of the leading term yields

$$\psi = -\sigma_c u_{\alpha}^\alpha, \quad (90)$$

and, neglecting the core extension energy, which is small, we arrive at the classical plate theory:

$$I = \int_{\Omega} \left[ 2\mu_s h_s \left( \sigma_s \left( \frac{u_{\alpha}^\alpha}{2} + u_{(\alpha,\beta)} u^{(\alpha,\beta)} \right) - P^\alpha u_{\alpha} \right) \right] d\Omega. \quad (91)$$

If $\beta > 4$, then the leading terms in $\psi$ are in the skin energy. Minimization over $\psi$ yields

$$\psi = 0.$$

The core extension energy is smaller than that of the skin. So, in the leading approximation we again arrive at the functional (91).

If $\beta = 4$, then the terms with $\psi$ in the core and the skin energy are of the same order, and, to find $\psi$, one has to solve a boundary value problem. However, the corresponding corrections to energy functional are of negligible order.
Therefore, the leading approximation is still given by the classical theory. The result holds for anisotropic plates as long as all eigenvalues of the elastic moduli tensor are of the same order.

8 Beams

All the above relations hold for plates. To get the corresponding relations for beams from three-dimensional elasticity the entire asymptotic analysis should be conducted again. We reduce the derivation of beam equations to the previous analysis by an additional assumption. Let $x^1 \equiv x$ be directed along the beam axis, $x^2$ and $x^3$ be the coordinates in the beam cross-section. We consider the cylindrical bending of the beam in $(x^1, x^3)$-plane. We assume that the stress components $\sigma_{21}, \sigma_{22}, \sigma_{23}$ and $\sigma_{33}$ are zero (more precisely, they are negligible in comparison with $\sigma_{11}$). Then the energy density becomes dependent only on three components of strains, $\varepsilon_{11}$ and $\varepsilon_{13}$. Denote this function by $F_1(\varepsilon_{11}, \varepsilon_{13})$. It is obtained by minimization of the true energy density over $\varepsilon_{21}, \varepsilon_{22}$ and $\varepsilon_{23}$:

$$F_1(\varepsilon_{11}, \varepsilon_{13}) = \min_{\varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}, \varepsilon_{33}} F(\varepsilon_{11}, \varepsilon_{13}, \varepsilon_{33}, \varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}).$$

We obtain

$$F_1(\varepsilon_{11}, \varepsilon_{13}) = \frac{1}{2} E \varepsilon_{11}^2 + \frac{1}{2} \mu \varepsilon_{13}^2,$$

where $E$ is the Young modulus in longitudinal direction. Computing the energy functional on the displacement fields derived above we obtain

$$I/w = \int \left[ \frac{E_s h_s^3}{12} u_{xx}^2 + \frac{E_s h_s}{4} \left( h_c \psi_x - h_s u_x \right)^2 + \frac{E_c h_c^3}{12} \psi_x^2 + \frac{\mu_c h_c}{2} \left( \psi + u_x \right)^2 - Pu \right] dx. \quad (92)$$

Here $\psi_1 \equiv \psi$ and we assume $P$ not depending on $y$.

9 Four-point bending of sandwich plates and beams

In this section we give a solution of the problem of four-point bending of sandwich plates within the universal asymptotic theory and compare it with the corresponding solution of classical theory and with some experimental
data. The sandwich layers are supposed to be isotropic. Notation is shown in Fig.6.

The in-plane coordinate $x^1$ is denoted further by $x$. The lateral displacement $u$ and rotation $\psi_1$ are functions of $x$ only while $\psi_2 \equiv 0$. We denote further $\psi_1$ by $\psi$. The functional of the asymptotic universal theory (31) divided by the plate length in $x^2$-direction, $w$, is:

$$I = \int_{-L}^{L} \left[ \frac{\mu_s h_s^3 (1 + \sigma_s)}{6} u_{xx}^2 + \frac{\mu_s h_s (1 + \sigma_s)}{2} \left( h_c \psi_{,x} - h_s u_{,xx} \right)^2 + \frac{\mu_c h_s^3 (1 + \sigma_c)}{12} \psi_{,x}^2 \right.$$

$$+ \left. \frac{\mu_c h_c}{2} (\psi + u_x)^2 - Pu \right] dx. \quad (93)$$

It is convenient to introduce the dimensionless displacement, $v = u/L$, the dimensionless coordinate $y = x/L$, relative thickness of the layers, $t = h_s/h_c$, dimensionless force per unit area, $p = PL/\mu_c h_c$, and the dimensionless coefficients,

$$A = \frac{\mu_s h_s^3 (1 + \sigma_s)}{3 \mu_c h_c L^2}, \quad B = \frac{\mu_s h_s (1 + \sigma_s) h_c^2}{\mu_c h_c L^2}, \quad C = \frac{h_c^2 (1 + \sigma_c)}{6 L^2}. \quad (94)$$

The energy functional in terms of dimensional variables takes the form:

$$I/\mu_c h_c L w = \int_{-1}^{1} \left[ \frac{1}{2} A v_{,yy}^2 + \frac{1}{2} B \left( \psi_{,y} - t v_{,yy} \right)^2 + \frac{1}{2} C y_{,y}^2 + \frac{1}{2} \left( \psi + v_y \right)^2 - pu \right] dy. \quad (95)$$

The minimization problem can be simplified by replacing $v(y)$ by the new required function $\theta(y) = v_y$. To see that we note that the total force acting on the system is zero,

$$\int_{-1}^{1} p dy = 0. \quad (96)$$
For any function $p(y)$ which obeys (96) there is a function $q(y)$ such that

$$p(y) = \frac{dq(y)}{dy}, \quad q(1) = q(-1) = 0.$$ 

Integrating by parts the last term in (95) we obtain the functional which depends only on $\psi(y)$ and $\theta(y)$:

$$I = \int_{1}^{1} \left[ \frac{1}{2} A \theta_{,yy} + \frac{1}{2} B \left( \psi_{,y} - t \theta_{,y} \right)^2 + \frac{1}{2} C \psi_{,yy} + \frac{1}{2} (\psi + \theta)^2 + q \theta \right] dy. \quad (97)$$

In four-point bending $q(y)$ is an odd function. Accordingly, the minimizing functions, $\psi(y)$ and $\theta(y)$, are odd as well. Therefore, the minimization problem for the functional (97) is equivalent to the minimization problem for the functional

$$I = \int_{0}^{1} \left[ \frac{1}{2} A \theta_{,yy} + \frac{1}{2} B \left( \psi_{,y} - t \theta_{,y} \right)^2 + \frac{1}{2} C \psi_{,yy} + \frac{1}{2} (\psi + \theta)^2 + q \theta \right] dy$$

on the set of all functions $\psi(y)$ and $\theta(y)$ defined on the segment $0 \leq y \leq 1$ and vanishing at the origin,

$$\psi(0) = \theta(0) = 0. \quad (98)$$

The Euler equations of this variational problem are:

$$\psi + \theta - \frac{d}{dy} \left[ A \theta_{,y} - B t \left( \psi_{,y} - t \theta_{,y} \right) \right] + q = 0, \quad A \theta_{,y} - B t \left( \psi_{,y} - t \theta_{,y} \right) \big|_{y=1} = 0; \quad (99)$$

$$\psi + \theta - \frac{d}{dy} \left[ C \psi_{,y} + B \left( \psi_{,y} - t \theta_{,y} \right) \right] = 0, \quad C \psi_{,y} + B \left( \psi_{,y} - t \theta_{,y} \right) \big|_{y=1} = 0. \quad (100)$$

Numerical solutions of this boundary value problem on commercial software like Mathematica face difficulties caused by small parameters at high derivatives. Fortunately, in the case under consideration we can get around such difficulties by solving the problem analytically. Indeed, deducting (100) from (99) we get

$$\frac{d}{dy} \left[ C \psi_{,y} + B \left( \psi_{,y} - t \theta_{,y} \right) - A \theta_{,y} + B t \left( \psi_{,y} - t \theta_{,y} \right) \right] + q(y) = 0. \quad (101)$$

We introduce a function, $r(y)$, as an antiderivative of $q(y)$ with zero boundary condition on the right end:

$$\frac{dr(y)}{dy} = q(y), \quad r(1) = 0.$$ 

Since for $y \geq 0$,

$$p(y) = p_0 (\delta(y - b) - \delta(y - a)), \quad q(y) = p_0 (\Theta(y - b) - \Theta(y - a)),$$
where $\Theta(y)$ is the step function ($\Theta(y) = 0$ if $y < 0$, and $\Theta(y) = 1$ for $y > 0$), we have

$$r(y) = r(y) = p_0(\Theta_1(y - b) - \Theta_1(y - a) + b - a).$$

(102)

Here $\Theta_1(y)$ is antiderivative of $\Theta(y):$

$$\Theta_1(y) = \begin{cases} y & y \geq 0 \\ 0 & y < 0 \end{cases}, \quad \frac{d\Theta_1(y)}{dy} = \Theta(y).$$

Note that the dimensionless force parameter $p_0$ relates to the total force acting on the top side of the plate, $F$, as

$$p_0 = \frac{F}{2\mu_c h_c w}.$$  

From (101)

$$\left[C \psi_{,y} + B \left(\psi_{,y} - t\theta_{,y}\right) - A\theta_{,y} + Bt \left(\psi_{,y} - t\theta_{,y}\right)\right] + r(y) = \text{const.}$$

The constant is equal to zero due to the boundary conditions at $y = 1$. Hence,

$$(C + B + Bt)\psi_{,y} - (A + Bt + Bt^2)\theta_{,y} + r(y) = 0.$$  

(103)

Let us introduce transverse shear $\varphi = \psi + \theta$. The derivatives $\psi_{,y}$ and $\theta_{,y}$ can be expresses in terms of $\varphi_{,y}$ from (103) and equation

$$\psi_{,y} + \theta_{,y} = \varphi_{,y}.$$  

We have

$$\theta_{,y} = m\varphi_{,y} + \frac{1}{\Delta} r(y), \quad \psi_{,y} = (1 - m)\varphi_{,y} - \frac{1}{\Delta} r(y).$$

(104)

where

$$\Delta = B_1 + B_2, \quad B_1 = B + C + Bt, \quad B_2 = A + Bt + Bt^2, \quad m = B_1/\Delta.$$  

Plugging (104) in (99) we obtain a boundary value problem for $\varphi(y)$:

$$\varphi - \kappa^2 \frac{d^2\varphi}{dy^2} + mp_0(\Theta(y - b) - \Theta(y - a)) = 0, \quad \varphi(0) = 0, \quad \varphi_{,y}\big|_{y=1} = 0.$$  

(105)

Here we introduced the notation

$$\kappa^2 = \frac{A(B + C) + BCt^2}{\Delta}.$$  

Consider an auxiliary boundary value problem,

$$g(y, b) - \kappa^2 \frac{d^2g(y, b)}{dy^2} + \Theta(y - b) = 0, \quad g(0, b) = 0, \quad \frac{dg(y, b)}{dy}\bigg|_{y=1} = 0.$$  

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Function $g(y, b)$ and its first derivative are continuous at $y = b$. The solution of this boundary value problem can be written explicitly:

$$g(y, b) = \begin{cases} -\sinh \frac{1-b}{\alpha} \sinh \frac{y}{\alpha} / \cosh \frac{1}{\alpha} & y \leq b \\ -1 + (\cosh \frac{1-y}{\alpha} \cosh \frac{b}{\alpha} / \cosh \frac{1}{\alpha}) & y \geq b \end{cases}$$

Then, obviously,

$$\varphi(y) = m p_0 (g(y, b) - g(y, a)).$$

Function $\theta(y)$ can be found from (104) and (102):

$$\theta(y) = m \varphi(y) + \frac{1}{\Delta} \int_0^y r(z) dz = m \varphi(y) + \frac{p_0}{\Delta} (\Theta_2(y - b) - \Theta_2(y - a) + (b - a)y),$$

where $\Theta_2(y)$ is antiderivative of $\Theta_1(y)$:

$$\Theta_2(y) = \begin{cases} y^2/2 & y \geq 0 \\ 0 & y < 0 \end{cases}, \quad \frac{d\Theta_2(y)}{dy} = \Theta_1(y).$$

The displacement is found from (107), (??) and the condition at the supporting point $v(b) = 0$:

$$v(y) = \int_b^y \theta dy = m^2 p_0 \left[ \int_b^y g(z, b) dz - \int_b^y g(z, a) dz \right]$$

$$+ \frac{p_0}{\Delta} (\Theta_3(y - b) - \Theta_3(y - a) + \frac{b - a}{2} (y^2 - b^2) + (b - a)^3/6).$$

Here $\Theta_3(y)$ is antiderivative of $\Theta_2(y)$:

$$\Theta_3(y) = \begin{cases} y^3/6 & y \geq 0 \\ 0 & y < 0 \end{cases}.$$

To compare this solution with the solution by classical plate theory we note that the functional of the classical theory is obtained from the functional (95) by setting $\psi = -v_{,y}$. We obtain

$$I/\mu_c h_c Lw = \int_{-1}^{1} \left[ \frac{1}{2} D v_{,yy}^2 - pv \right] dy.$$  

where the dimensionless bending rigidity is

$$D = A + B(1 + t)^2 + C = \Delta.$$

Functional (109) in terms of $\theta(y) = du/dy$ for odd $\theta(y)$ becomes

$$I/2\mu_c h_c Lw = \int_0^1 \left[ \frac{1}{2} D \theta_{,y}^2 + q\theta \right] dy.$$
Here minimum is sought on the set of all functions \( \theta(y) \) which are zero at \( y = 0 \). The minimizer is the solution of the boundary value problem

\[
D \frac{d^2 \theta}{dy^2} = p_0(\Theta(y - b) - \Theta(y - a)), \quad \theta(0) = 0, \quad \theta'(1) = 0.
\]

The solution is

\[
D \theta = p_0(\Theta_2(y - b) - \Theta_2(y - a) + (b - a)y).
\]

For the displacement, vanishing at \( y = b \), we have

\[
u(y) = \int_b^u \theta dy = \frac{p_0}{D}(\Theta_3(y - b) - \Theta_3(y - a) + (b - a)(y^2/2 - b^2/2) + (b - a)^3/6).
\]

The ratio, \( R \), of the slopes of the load-displacement curves in classical and the asymptotic theories is

\[
R = \frac{u(a)}{v(a)}.
\]

Comparing (108) and (110) we see that the entire difference between displacements computed by the classical and the asymptotic theories is caused by the transverse shear \( \varphi(y) \).

The displacements found by classical and asymptotic universal theory are shown in Fig.7 for hard-skin thin plates \( h_s/h_c = 0.05, \Lambda = 0.01 \) for two values of \( h/l, h/l = 0.1 \) (Fig.7a) and \( h/l = 0.01 \) (Fig.7b). The corresponding values of \( \alpha = \log \Lambda/\log(h/l) \) are 2 and 1. The values of \( a \) and \( b \) are chosen as \( a = 0.2, b = 0.6 \). The displacements are referenced to the force, \( F \), acting to the top side of the plate, i.e. it is plotted the dimensionless function

\[
\frac{2\mu_s h_c w}{FL} u(y).
\]

It is seen that for the plate of moderate thickness 0.1, the displacements by classical and asymptotic theories differ considerably bringing a large difference in the slope of the load-displacement curves; the slope ratio is equal to the ratio of the dimensionless displacement values at \( y = 0.2 \). For thin plate, \( h/l = 0.01 \), the results of classical and asymptotic theory are practically indistinguishable. This is in full compliance with Fig.2: we see from this Figure that for \( h/l = 0.01 \) and \( \alpha = 1 \) the ratio of the slopes is only slightly differs from 1.

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**References**
Fig. 7. The dimensionless displacements found by classical theory (upper curve, a) and asymptotic universal theory (bottom curve, a) for moderate thickness, \( h/l = 0.1 \). For thin plates, \( h/l = 0.01 \), the dimensionless displacements found by classical theory and asymptotic universal theory are practically indistinguishable (b).


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