TRANSFER FUNCTION MODELING OF DISTRIBUTED PIEZOELECTRIC VIBRATION ENERGY HARVESTERS

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ABSTRACT

Extensive research has been conducted on vibration energy harvesting utilizing a distributed piezoelectric beam structure. A fundamental issue in the design of these harvesters is the understanding of the response of the beam to arbitrary external excitations (boundary excitations in most models). The modal analysis method has been the primary tool for evaluating the system response. However, a change in the model boundary conditions requires a reevaluation of the eigenfunctions in the series and information of higher-order dynamics may be lost in the truncation. In this paper, a frequency domain modeling approach based in the system transfer functions is proposed. The transfer function of a distributed parameter system contains all of the information required to predict the system spectrum, the system response under any initial and external disturbances, and the stability of the system response. The methodology proposed in this paper is valid for both self-adjoint and non-self-adjoint systems, and is useful for numerical computer coding and energy harvester design investigations. Examples will be discussed to demonstrate the effectiveness of this approach for designs of vibration energy harvesters.

1 INTRODUCTION

Over the past decade, many innovations, particularly in the area of MEMS research, have made it possible to design micro sensors and actuators which are able to operate with very low power requirements [1], thus allowing sensor networks to be deployed and operated in an autonomous manner [2]. Many of these sensors can be used in applications with self-contained power sources packaged with the sensors. While batteries are currently still used in many of these applications, piezoelectric vibration energy harvesters have shown incredible potential for use with self-powered devices in applications where ambient vibration energy is present. The development of self-powered sensors and devices not only reduce environmental hazards related to battery disposals, it also presents new opportunities and challenges in the emerging research field of renewable energy sources for sensors and electronic systems.

The most common sources of vibration energy range from harnessing pressure variations in pipe flow, acoustical energy in mobile phones, flutter on airplane wings, vibration of rotational machinery, packages on a conveyor belt, or vehicle motion on bridges, with potential applications to the developments of self-powered sensors for material fatigue detection, structural health monitoring of highway bridges, active damping devices, and surveillance monitoring [3]-[6]. Some feasibility studies have shown that the ambient energy levels in many structural applications experiencing wind and traffic loads are sufficient to power simple wireless sensors [6]. Interests in developing reliable, inexpensive self-powered sensors have led to extensive research in energy harvesting using piezoelectric materials in recent years. A comprehensive review of this research through 2006 can be found in [7], and an overview of piezoelectric energy harvesting for MEMS devices is presented in [3], [8].

Theoretical analysis of vibration energy harvesters involves the mechanical modeling of the vibrator itself, the modeling of electro-mechanical coupling via the piezoelectric material and the electrical modeling of the circuit. The modeling of the mechanical vibrator varies in complexity and technique. Analytical models have taken a number of different forms, ranging from the simplistic single-degree-of-freedom (SDOF) systems to complex computational methods. In most cases, the mechanical vibrator is modeled as a cantilever beam which may include a proof mass at the tip. This geometry increases the displacement for a given package size and reduces the complexity of the mechanical modeling, however alternate geometries have also been investigated [9]-[10]. Recently, a comparison of a SDOF model with Euler-Bernoulli predictions shows the inaccuracy of oversimplifying the mechanical modeling of the vibrator [11]. In addition to analytical models,
computational methods such as finite elements methods [12] and equivalent circuit models [13] have been developed for use with harvester circuit analyses and designs.

Of greater interest than the mechanical modeling alone are the coupled models relating the electrical power generated to the vibration input. Novel modeling techniques for the coupled electro-mechanical system have been available for over a decade [14]. A comparison of the electrical output modeled using an uncoupled analysis, an in-phase analysis, and an analytic analysis is given in [15]. These coupled electro-mechanical models can then be used to evaluate the effective energy conversion of the entire system in order to evaluate the performance of the energy harvesters [16]-[17]. In conjunction with modeling, quite a few researchers have attempted to verify current results using laboratory measurements from prototype vibrators, including both macro scale cantilever beams [18]-[19] and MEMS devices [4],[20]. These experiments show that vibration harvesters which are properly tuned to the input frequency perform in a manner similar to that predicted by conventional modeling.

Although SDOF modeling of the vibration energy harvester provides a simple, closed-form solution that may be useful for design purpose, it was shown that significant errors could result [11]. A distributed parameter electromechanical cantilever beam model was recently studied in [21]. A modal analysis approach was employed to analyze the response of the beam to arbitrary base excitations. This approach, while useful, has several drawbacks. First, the response is expressed by an infinite series of eigenfunctions (usually those of an approximate, simpler structure), and these eigenfunctions may be difficult to obtain, particularly for systems with non-proportional damping (non-self-adjoint systems). Second, truncation of the series has to be made in numerical computations. This, along with inaccuracy in estimated eigenfunctions, can result in large numerical error, especially when the frequency is near the system resonances. Third, the series truncation effectively reduces the order of the model, which can lead to loss of information about the higher-order dynamics which may be important for the purposes of control and design of the energy harvesters. It should be noted that most of the modeling techniques referenced above utilizes time domain methodologies, focusing on the evaluation of the steady-state response. While most time domain modeling appears to be effective, the analysis is relatively complex, even for simple geometries such as a unimorph cantilever beam, and requires numerous assumptions such as the damping proportionality in order to be solved.

In this paper, we propose a frequency domain approach to investigating the response and design of vibration energy harvesters based on the concept of distributed transfer function. The transfer function of a distributed parameter system contains all of the information required to predict the system spectrum, the system response under any initial and external disturbances, and the stability of the system response [22]. This formulation has several important advantages over classical time domain methods: (i) the methodology is amenable to efficient numerical coding [23]; (ii) it can be employed to model non-self-adjoint systems subjected to arbitrary excitations; (iii) system response and its derivatives (such as strains) can be obtained simultaneously from the calculations; (iv) the formulation leads to compact formulas [24] and can easily incorporate well-known feedback control principles (such as the root locus method) for design purposes. This paper is organized as follows. Formulation of a distributed parameter in the frequency domain and derivation of the system transfer function are first described. The proposed methodology is then applied to the modeling of vibration energy harvesters with discussion on design strategies in the frequency domain.

2 TRANSFER FUNCTION OF A DISTRIBUTED PARAMETER ELEMENT

In this section, we outline the basic results for the derivation of the transfer function of a distributed parameter model [23]. Consider the response \( w(x,t) \) of a one-dimensional distributed parameter element subjected to an excitation \( f(x,t) \)

\[
\begin{bmatrix}
A & \frac{\partial^2}{\partial t^2} + B \frac{\partial}{\partial t} + C
\end{bmatrix} \begin{bmatrix} w(x,t) \\ \frac{\partial w(x,t)}{\partial t} \end{bmatrix} = f(x,t), \quad x \in (0,1), \quad t > 0
\]

with the inhomogeneous boundary conditions (such as base excitations for vibration harvesters)

\[
M_j w(x,t) \bigg|_{t=0} + N_j w(x,t) \bigg|_{t=0} = \gamma_{j0}(t), \quad j = 1,2,\ldots,n
\]

where \( M_j \) and \( N_j \) are temporal-spatial, linear differential operators of the proper order, and \( \gamma_{j0}(t) \) are known functions.

In (1), \( A, B \), and \( C \) are spatial differential operators of the form

\[
A = \sum_{k=0}^{n} a_k \frac{\partial^k}{\partial x^k}, \quad B = \sum_{k=0}^{n} b_k \frac{\partial^k}{\partial x^k}, \quad C = \sum_{k=0}^{n} c_k \frac{\partial^k}{\partial x^k}
\]

which are associated with the inertia; damping, Coriolis acceleration and mass transport; and stiffness, centrifugal forces, and circulatory effects, respectively. The initial conditions are given by

\[
w(x,t) \bigg|_{t=0} = u_0(x), \quad \frac{\partial}{\partial t} w(x,t) \bigg|_{t=0} = v_0(x), \quad x \in (0,1).
\]

Laplace transform of (1) to the frequency domain leads to

\[
(s^2 A + sB + C) \bar{w}(s,x) = \bar{f}(s,x)
\]

where, \( \bar{w}(s,x) \) and \( \bar{f}(s,x) \) are the Laplace transforms of \( w(x,t) \) and \( f(x,t) \), respectively, and \( s \) is the complex frequency parameter. Transform of the boundary conditions (2) gives

\[
\bar{M}_j \bar{w}(s,x) \bigg|_{x=0} + \bar{N}_j \bar{w}(s,x) \bigg|_{x=1} = \bar{\gamma}_j(s), \quad j = 1,2,\ldots,n
\]

where \( \bar{M}_j \) and \( \bar{N}_j \) are the operators \( M_j \) and \( N_j \) with the time-derivative operators \( \frac{\partial}{\partial t} \) and \( \frac{\partial^2}{\partial t^2} \) replaced by \( s \) and \( s^2 \).
respectively, $\tau_{y}(s)$ is the Laplace transform of $\gamma_{y}(s)$, and $\tau_{x}(s)$ is a polynomial of $s$ representing the initial conditions at the boundaries $x = 0$ and $1$, respectively. Symbolically, the solution of (5) and (6) can be written in the form [25]

$$\tilde{w}(x, s) = \int_{0}^{1} W_{x}(x, \xi, s) \tilde{f}_{d}(\xi, s) d\xi + \sum_{j=1}^{n} \phi_{j}(x, s) \tau_{j}(s)$$  \hspace{1cm} (7)

where the integral kernel, $W_{x}(x, \xi, s)$, is the transfer function of the distributed parameter system (1), and the boundary transfer function $\phi_{j}(x, s)$ represent the influence of the inhomogeneity of the boundary conditions (6) on the system response. It is noted that the inverse Laplace transfer of $W_{x}(x, \xi, s)$ is the Green's function of the distributed parameter system (1).

To obtain the transfer function of the distributed parameter system, we first cast the problem in the state-space form. Set

$$q(x, s) = \begin{bmatrix} \tilde{w}(x, s) \\ \frac{\partial}{\partial \xi} \tilde{w}(x, s) \\ \ldots \\ \frac{d^{n-1}}{d \xi^{n-1}} \tilde{w}(x, s) \end{bmatrix} \in \mathbb{C}^{n}$$  \hspace{1cm} (8a)

$$f(x, s) = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \frac{\partial}{\partial \xi} f_{d}(x, s) \\ \ldots \\ \frac{d^{n-1}}{d \xi^{n-1}} f_{d}(x, s) \end{bmatrix} \in \mathbb{C}^{n}$$  \hspace{1cm} (8b)

$$\gamma(s) = \begin{bmatrix} \tau_{1}(s) \\ \tau_{2}(s) \\ \ldots \\ \tau_{n}(s) \end{bmatrix} \in \mathbb{C}^{n}$$  \hspace{1cm} (8c)

$$A(s) = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \in \mathbb{C}^{n \times n}$$

$$B(s) = \begin{bmatrix} -d_{1}(s) \\ -d_{2}(s) \\ \ldots \\ -d_{n}(s) \end{bmatrix} \in \mathbb{C}^{n \times 1}$$

where the coefficients of the matrix $B(s)$ are

$$d_{i}(s) = \frac{a_{i} \xi^{2} + b_{i} \xi + c_{i}}{a_{i} \xi^{2} + b_{i} \xi + c_{i}}, \quad k = 0, 1, \ldots, n - 1$$  \hspace{1cm} (8d)

Note that the matrix $F(s)$ is not a function of $x$ for uniformly distributed parameter elements. By the definitions of (8), equations (5) and (6) are cast into a matrix form

$$\frac{\partial}{\partial \xi} q(x, s) = A(s) q(x, s) + f(x, s), \quad x \in (0, 1)$$  \hspace{1cm} (9a)

$$M(s) q(0, s) + N(s) q(1, s) = \gamma(s)$$  \hspace{1cm} (9b)

where $M(s)$ and $N(s)$ are $n \times n$ complex matrices, consisting of the coefficients of the operators $\tilde{M}_{j}$ and $\tilde{N}_{i}, \quad j = 1, 2, \ldots, n$. A compact formula can be derived for the solution of problem (9a, b).

$$q(x, s) = \int_{0}^{1} G(x, \xi, s) f_{d}(\xi, s) d\xi + H(x, s) \gamma(s), \quad x \in (0, 1)$$  \hspace{1cm} (10)

where the matrix Green's function written in terms of the fundamental matrix of (9a, b), $G(x, \xi, s) = e^{e^{(s)}}$, is

$$G(x, \xi, s) = \begin{bmatrix} \Phi(x, s) \left( M(s) + N(s) e^{(s)} \right)^{-1} & M(s) \Phi(-\xi, s), & \xi < x \\ \Phi(x, s) \left( M(s) + N(s) e^{(s)} \right)^{-1} N(s) \Phi(\xi - 1, s), & \xi > x \end{bmatrix}$$  \hspace{1cm} (11)

and the boundary transfer function is

$$H(x, s) = e^{(s)} \left( M(s) + N(s) e^{(s)} \right)^{-1}$$  \hspace{1cm} (12)

It is noted that the solution (10) gives the response and its derivatives (such as the strain of a beam) simultaneously. Write the matrix transfer functions as

$$G(x, \xi, s) = \begin{bmatrix} g_{11}(x, \xi, s) & \cdots & g_{1n}(x, \xi, s) \\ \vdots & \ddots & \vdots \\ g_{m1}(x, \xi, s) & \cdots & g_{mn}(x, \xi, s) \end{bmatrix}$$  \hspace{1cm} (13a)

$$H(x, s) = \begin{bmatrix} h_{1}(x, s) \\ \vdots \\ h_{m}(x, s) \end{bmatrix}$$  \hspace{1cm} (13b)

By (8), (10) and (13), it is seen that

$$\tilde{w}(x, s) = \int_{0}^{1} g_{mn}(x, \xi, s) \tilde{f}_{d}(\xi, s) d\xi + \sum_{j=1}^{n} h_{j}(x, s) \tau_{j}(s)$$  \hspace{1cm} (14)

The distributed transfer function and the boundary transfer functions of the system are thus given by

$$W_{d}(x, \xi, s) = \frac{g_{mn}(x, \xi, s)}{a_{n} s^{2} + b_{n} s + c_{n}}$$  \hspace{1cm} (15a)

$$\phi_{j}(x, s) = \frac{h_{j}(x, s)}{a_{n} s^{2} + b_{n} s + c_{n}}$$  \hspace{1cm} (15b)

### 3 COMPUTATIONS OF THE SYSTEM RESPONSE

Equations (11), (12) and (15) suggest a new method for evaluation of the system transfer function. To determine $W_{d}(x, \xi, s)$ and $\phi_{j}(x, s)$, one only needs to calculate the fundamental matrix $e^{(s)}$ and $\left( M(s) + N(s) e^{(s)} \right)^{-1}$. These two matrices, with low order in general (for example, for vibration and wave propagation in continua, usually $n \leq 4$), can be accurately calculated. Note that $G(x, \xi, s)$ and $H(x, s)$ in (11) are expressed in closed-form. Therefore, $W_{d}(x, \xi, s)$ and $\phi_{j}(x, s)$ can be precisely evaluated in closed-form.

The numerical scheme for the evaluation of $W_{d}(x, \xi, s)$ and $\phi_{j}(x, s)$ is as follows:

i) For a given system (1) and boundary conditions (2), form the matrices $A(s)$, $M(s)$, and $N(s)$, which are very simple to model;

ii) For given $x$ and $s$, evaluate the matrices $e^{(s)}$ and $\left( M(s) + N(s) e^{(s)} \right)^{-1}$; and

iii) Calculate $W_{d}(x, \xi, s)$ and $\phi_{j}(x, s)$ based on (11), (12), and (15).

Remarks on the computations and evaluations of the system response and derivatives (e.g., strain) for harvester designs:

1) There are many methods that give exact presentations of $\Phi(x, s)$ ([26], [27]). However, because of the special form of
\[ A(s), M(s), N(s) \] matrix diagonalization may be simpler to use. With advances in computational tools such as MATLAB, these computations can be executed with ease and precision. The incorporation of control principles using MATLAB toolboxes makes this methodology amenable to numerical coding for more complex problems in the designs of energy vibration harvesters.

2) The proposed method is exact; no truncation or approximation has been made. It requires no knowledge of the system eigensolutions, and the method is valid for both self-adjoint and no-self-adjoint systems.

3) The above numerical scheme is convenient in computer coding. The system type, differential operators, and boundary conditions are treated uniformly through easy assignment of the matrices \( A(s), M(s), \) and \( N(s). \) Because \( A(s) \) is independent of \( M(s) \) and \( N(s) \), changes in the boundary conditions do not require recalculation of the fundamental matrix \( e^{M(s)} \). Indeed, changes in boundary conditions require only minor changes in the definitions of \( M(s) \) and \( N(s) \) and no modification in the computation algorithms.

4) The characteristic equation of the distributed parameter system (1), according to (11) and (12), is

\[ \det[M(s) + N(s)e^{A(s)}] = 0 \quad (16) \]

whose roots are the eigenvalues of the distributed parameter system, or the poles of \( W_o(x, \xi, s) \) and \( \phi_j(x, s) \).

5) Although this paper is focused on application to distributed piezoelectric models with constant parameters, the approach discussed can be extended to distributed systems with parameters as functions of the spatial coordinate \( x \), i.e., \( a_i = a_i(x), \ b_i = b_i(x), \) and \( c_i = c_i(x). \) In this case, the matrix formulation for \( W_o(x, \xi, s) \) and \( \phi_j(x, s) \) is still valid in terms of a fundamental matrix \( \Phi(x, s). \) The closed-form of \( \Phi(x, s) \) for general distributed parameter systems is usually difficult to obtain, although approximation methods are available [28]. However, for a non-uniformly distributed parameter system, exact determination of its eigensolutions is also difficult (and may be more difficult). So, the eigenfunction expansion series representation of the system transfer function may not be appropriate at all.

It is noted that the frequency response of the distributed parameter model can easily be obtained. Consider the case of homogeneous boundary conditions and a harmonic point force of unit amplitude applied at \( \xi = \xi_0 \). The steady-state response, in closed-form by replacing \( s \) by \( \omega \), is simply

\[ w(x, t) = W_o(x, \xi_0, \omega)e^{i\omega t} \quad (17) \]

General formulas and procedures for evaluating the transient response can be found in [29]. The evaluation of the mode shapes and strain distribution can also be obtained by solving the associated eigenvalues problem [23], [30].

### 4. PIEZOELECTRIC ENERGY HARVESTER MODEL

#### 4.1 Mechanical Beam Model

The frequency domain method is applied to a distributed parameter model of unimorph vibration energy harvester described in [21]. The model is a uniform composite Euler-Bernoulli beam consisting of a PZT layer bonded to the substrate layer, where the electrical feedback from the harvesting circuit is not considered in the mechanical modeling. The boundary conditions are standard for a cantilever beam with harmonic base excitation (translation with small rotation) at one end, see Figure 1. The governing equation of motion of the uncoupled beam can be expressed as:

\[
m \frac{d^2w(x, t)}{dt^2} + c_o \frac{dw(x, t)}{dt} + c_I \frac{d^2w(x, t)}{dx^2} + EI \frac{d^4w(x, t)}{dx^4} = 0 \quad (18a)
\]

where \( w_b(x, t) \) is the base excitation, and \( w(x, t) \) the response of the beam relative to the base excitation; the total displacement is: \( W(x, t) = w_b(x, t) + w(x, t) \). The parameters are defined as: \( m \) the mass per unit length, \( c_s \) the equivalent strain rate damping, \( c_v \) the viscous air damping coefficient, and \( EI \) the product of the Young's modulus and moment of inertia. In Fig. 1, \( y(t) \) and \( \theta(t) \) are the linear and angular boundary excitations applied at \( L = 0 \).

![Figure 1. Schematic of a unimorph vibration energy harvester under base excitations (sources of vibration energy).](image)

Normalizing the spatial domain results in:

\[
\left(m \frac{d^2}{dt^2} + c_o \frac{dx}{dt} + c_I \frac{d^2}{dx^2} + EI \frac{d^4}{dx^4}\right)w(x, t) = f(x, t) \quad (18b)
\]

where,

\[ x = X L \quad \frac{\partial}{\partial X} = \frac{\partial}{\partial x} \frac{X}{L} \quad \frac{1}{L} \frac{\partial}{\partial x} \]

There are two different formulations to the basic mechanical modeling problem.

(i) **Base excitation treated as external excitation**

By considering the excitation at the base as an external displacement, the state-space formulation of (8a-8e) results in:
The distributed transfer function of the frequency response of the piezoelectric energy harvester model can be obtained using the computation procedure outlined in sections 2 and 3. In particular, once the system transfer function is obtained, equation (17) can be used to evaluate the system response to harmonic input excitations. The harmonic steady-state response is then given by:

\[
\tilde{w}(x,s) = \frac{\int \frac{g_{ds}(x,\xi,s)L^2}{c_{ls} + EI} \tilde{f}_{so}(\xi,s)d\xi}{s + \frac{1}{L^2}} + y(x,t) + xL \cdot h(t)
\]

where

\[
\tilde{f}(x,s) = \mathcal{L}\left[-mi\tilde{w}_0\right]
\]

and

\[
g(t) = Y_s \cdot e^{i\omega t} \quad \text{and} \quad h(t) = \theta_s \cdot e^{i\omega t}
\]

(ii) Base Excitation Treated as Boundary Excitation

An alternate technique involves treating the excitation of the base as a boundary excitation rather than an external excitation. When this method is employed, the boundary conditions of the state-space formulation are modified to include the excitation at the boundary:

\[
\gamma(s) = \begin{bmatrix} Y_0 \\ \theta_0 \\ 0 \\ 0 \end{bmatrix}
\]

and the forcing function \( \tilde{f}(x,s) \) is reduced to zero. This formulation results in a much simpler solution to the stated problem, and produces modeling identical to that found using the external excitation. For instance, when \( \theta_0 = 0 \),

\[
w(x,t) = \left[H_{11}(x,\omega) - 1\right]Y_0 e^{i\omega t}
\]

Modeling Implementation

For comparison, the cantilever dimensions and material properties are taken from [21] as follows:

**Table 1: Dimensional and Material Properties of Cantilever Beam**

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (m)</td>
<td>( L = 0.01 )</td>
</tr>
<tr>
<td>Width (m)</td>
<td>( b = 0.002 )</td>
</tr>
<tr>
<td>Thickness (m)</td>
<td>( hs = 5 \times 10^{-4} )</td>
</tr>
<tr>
<td>Thickness (m)</td>
<td>( hp = 4 \times 10^{-4} )</td>
</tr>
</tbody>
</table>

**Material Properties**

- Mass density of substructure (kg/m³) \( \rho_{s} = 7165 \)
- Mass density of PZT (kg/m³) \( \rho_{p} = 7800 \)
- Substructure Young's modulus (Pa) \( Y_s = 100 \times 10^9 \)
- PZT Young's modulus (Pa) \( Y_p = 66 \times 10^9 \)

The following values were derived from these properties:

**Table 2: Derived Values for the Cantilever Beam**

- Linear mass (kg/m) \( m = \rho_{s} b h_s + \rho_{p} b h_p \)
- Young's modulus for composite material \( Y = \frac{Y_s h_s}{h_s + h_p} + \frac{Y_p h_p}{h_s + h_p} \)
- Area moment of inertia \( I = \frac{1}{12} b (h_s + h_p)^3 \)
- Air damping coefficient \( c_a = 4.886 \times m \)
- Strain-rate damping coefficient \( c_s = 1.2433 \times 10^{-5} \times Y \)

Implementation of his method applied to the cantilever beam follows the following procedure:

1. Input \( A, M, N, F \) and \( \gamma \) based on system equation and boundary conditions
2. Calculate \( H(x,s) \) and \( G(x,\xi,s) \) using the fundamental matrix \( \Phi(x,s) = e^{A(s)x} \)
3. Determine \( W_j(x,\xi,s) \) and \( \phi_j(x,s) \)
4. Numerically integrate \( \int_{0}^{1} g_{jn}(x,\xi,s) W_j(x,\xi,s) d\xi \)
5. and solve for \( \tilde{w}(x,s) = \int_{0}^{1} g_{jn} W_d d\xi + \sum_{j=1}^{n} h_j \tilde{f}_j \)

Figure 2: Flowchart of numerical implementation of transfer function method
This implementation is straight forward, and is coded in general subroutines which can be imbedded into larger system modeling or design optimization programs.

**Results**

The results from the presented method correlate well with previously developed models. Figures 3 and 4 show the frequency response function of the cantilever beam tip displacement using the transfer function approach compared with modal analysis modeled by [21]. As can be seen, the methods are identical except near the anti resonance frequency, 551 Hz. Closer investigation of this area, as seen in Figure 5, shows that additional terms in the modal analysis should converge to the transfer function method results developed here for the undamped beam model.

![Figure 3: Uncoupled FRF for undamped cantilever beam](image1)

![Figure 4: Uncoupled FRF for damped cantilever beam](image2)

The presented formulation and accompanying plots show that the frequency response can be obtained in a straightforward manner using frequency domain analysis. This type of analysis presents great potential for for harvester design optimization in the frequency domain.

**Notes on the fundamental characteristics of the transfer function model of the cantilever beam:**

Recall from eq. (16) the characteristic equation of the system:

\[ \det (M(s) + N(s)e^{\alpha(s)}) = 0 \]

which when evaluated using (19) results in

\[ 1 + \cosh(\tilde{\beta}) \cos(\tilde{\beta}) = 0 \]

For the undamped system, \( \tilde{\beta}^u = -\frac{ms^2}{EI} \), when evaluated at \( s = i \omega \) gives the natural frequencies \( \omega_n = \tilde{\beta}^u \sqrt{\frac{EI}{mL^4}} \), which is identical to that found in [21].

For the damped system, \( \tilde{\beta}^d = -\frac{ms^2 + c_s s}{c_s I s + EI} L^4 \), which when rewritten as:

\[ ms^2 L^4 + (\tilde{\beta}^d c_s I + c_s L^4) s + \tilde{\beta}^d EI = 0 \]

can be compared with the standard form resulting in:

\[ \zeta = \frac{c_s \omega_n^2 + \omega_n}{2E} \frac{1}{2m \omega_n} \]

\[ s = -\zeta_s \omega_n \pm i \omega_n \sqrt{1 - \zeta_s^2} \]

These results indicate that the roots of the the characteristic equation for both the damped and undamped cases are identical to those found using modal analysis which is confirmed by the comparison shown above.
4.2 Coupled Electro-Mechanical Beam Model

The frequency domain method will now be applied to a distributed parameter model of unimorph vibration energy harvester where the electrical feedback from the harvesting circuit is considered. The boundary conditions are the same as those for the uncoupled cantilever beam. The governing equation of motion of the coupled beam (normalized in the spatial domain) can be expressed as:

\[
m \frac{\partial^2 w(x,t)}{\partial t^2} + c_a \frac{\partial w(x,t)}{\partial t} + \frac{c_f L}{L'} \frac{\partial^3 w(x,t)}{\partial x^3} + \frac{E I}{L'} \frac{\partial^4 w(x,t)}{\partial x^4} + \varphi(v(t)) = 0
\]

Taking the Laplace transform gives:

\[
m \frac{\partial^2 \tilde{w}(s,x)}{\partial t^2} + c_a \frac{\partial \tilde{w}(s,x)}{\partial t} + \frac{c_f L}{L'} \frac{\partial^3 \tilde{w}(s,x)}{\partial x^3} + \frac{E I}{L'} \frac{\partial^4 \tilde{w}(s,x)}{\partial x^4} + \tilde{\varphi}(s) = 0
\]

Following [21], the viscous air damping term, \( c_a \frac{\partial \tilde{w}(s,x)}{\partial t} \) will be neglected. Taking the Laplace transform gives:

\[
m \frac{\partial^2 \tilde{w}(s,x)}{\partial t^2} + \frac{c_f L}{L'} \frac{\partial^3 \tilde{w}(s,x)}{\partial x^3} + \frac{E I}{L'} \frac{\partial^4 \tilde{w}(s,x)}{\partial x^4} + \tilde{\varphi}(s) = 0
\]

In the uncoupled mechanical modeling, it was shown that the base excitations can be treated either as an external excitation to the equations of motion for \( w(x,s) \) or as a boundary term for the equations of motion for \( w(x,s) \). Both formulations will be discussed in this section to show the numerical efficacy of the transfer function method.

Base Excitation Treated as External Excitation

The \( \mathbf{A}(s), \mathbf{M}(s), \mathbf{N}(s), \) and \( \varphi(s) \) matrices of the state space formulation remain identical to those used in the uncoupled system equations (19a-c), and as before, the beam response is:

\[
\tilde{w}(s,x) = \int_0^1 G_{\mathbf{n}}(x,\xi,s) \tilde{f}(\xi,s) L' d\xi
\]

The forcing function term contains the electrode voltage term which provides the electro-mechanical coupling as such:

\[
\tilde{f}(x,s) = \mathbf{f}(x,s) - \varphi(v(t)) \left[ \frac{d \delta(x)}{dx} - \frac{d \delta(x-L)}{dx} \right]
\]

The resulting beam response is:

\[
\tilde{w}(s,x) = \int_0^1 G_{\mathbf{n}}(x,\xi,s) \left[ \tilde{f}(\xi,s) - \mathbf{f}(\xi,s) - \varphi(v(t)) \left[ \frac{d \delta(x)}{dx} - \frac{d \delta(x-L)}{dx} \right] \right] L' d\xi
\]

Next, the voltage equation must be investigated. Based on the piezoelectric constitutive equations, the relationship between the beam displacement and the voltage output is as follows:

\[
\frac{dv}{dt} + \varphi(v(t)) = \int_0^1 \frac{d_3}{\varepsilon_s^3} d_1 Y h h_s \frac{\partial^3 w(x,t)}{\partial x^3} dX
\]

which when spatially normalized becomes:

\[
\frac{dv}{dt} + \varphi(v(t)) = \int_0^1 \frac{d_3}{\varepsilon_s^3} d_1 Y h h_s \frac{\partial^3 \tilde{w}(s,x)}{\partial x^3} dX
\]

taking the Laplace transform, and assuming that there are zero initial conditions, results in:

\[
\left( s + \frac{1}{\tau_e} \right) \tilde{v}(s) = -\int_0^1 \frac{d_3}{\varepsilon_s^3} d_1 Y h h_s \frac{\partial^3 \tilde{w}(s,x)}{\partial x^3} dX
\]

Solving for the voltage,

\[
\tilde{v}(s) = -\frac{\tau_e}{s + \frac{1}{\tau_e}} \left[ \frac{d_3}{\varepsilon_s^3} d_1 Y h h_s \frac{\partial^3 \tilde{w}(s,x)}{\partial x^3} dX \right]_{-1}^{1}
\]

the relationship between voltage and strain is clearly defined.

Based on the state space formulation of the transfer function, results can be found using:

\[
\frac{\varphi(s)}{\tilde{v}(s)} = \int_0^1 \frac{g_{ss}(x,\xi,s) \left[ \tilde{f}(\xi,s) - \mathbf{f}(\xi,s) - \varphi(v(t)) \left[ \frac{d \delta(\xi)}{d\xi} - \frac{d \delta(\xi-L)}{d\xi} \right] \right] L'}{c_e \varepsilon_s^3 + s \varepsilon_e} \frac{d\xi}{d\xi}
\]

where, \( g_{ss}(x,\xi,s) \), is a component of the Green’s Function, \( \mathbf{G}(x,\xi,s) \), defined in equation (11). Evaluating (29) for \( x = 1 \) and substituting the result into (30) results in a closed-form solution for \( \tilde{v}(s) \).

Base Excitation Treated as Boundary Excitation

In treating the base excitation as a boundary condition, the spatially normalized governing equation of motion becomes:

\[
m \frac{\partial^2 \tilde{w}(s,x)}{\partial t^2} + c_a \frac{\partial \tilde{w}(s,x)}{\partial t} + \frac{c_f L}{L'} \frac{\partial^3 \tilde{w}(s,x)}{\partial x^3} + \frac{E I}{L'} \frac{\partial^4 \tilde{w}(s,x)}{\partial x^4} + \varphi(s) = 0
\]

Where the boundary conditions conditions are:

\[
w(0,t) = g(t), \quad \frac{1}{L} \frac{dw}{dx}(0,t) = h(t),
\]

\[
\frac{1}{L'} \frac{d^2 w}{dx^2}(1,t) = 0, \quad \frac{1}{L'} \frac{d^3 w}{dx^3}(1,t) = 0
\]

Consider the case where, \( \theta_e = 0 \), and hence the slope of the total displacement and the relative displacement are the same:

\[
\frac{\partial W(x,s)}{\partial x} = \tilde{w}(s,x).
\]

From the previous formulation, frequency domain voltage and displacement slope equations become:

\[
\tilde{v}(s) = -\frac{\tau_e}{s + \frac{1}{\tau_e}} \left[ \frac{d_3}{\varepsilon_s^3} d_1 Y h h_s \frac{\partial \tilde{W}(s,x)}{\partial x} \right]_{-1}^{1}
\]

\[
\frac{\varphi(s)}{\tilde{v}(s)} = \int_0^1 \frac{g_{ss}(x,\xi,s) \left[ \frac{d \delta(\xi)}{d\xi} - \frac{d \delta(\xi-L)}{d\xi} \right] L'}{c_e \varepsilon_s^3 + s \varepsilon_e} \frac{d\xi}{d\xi}
\]
which when combined together, provide an alternate formulation of the solution for the electrode voltage, \( V(s) \).

5. CONCLUSION

In this paper, we have proposed an exact, closed-form semi-analytical evaluation of the response of a piezoelectric vibration energy harvester using a distributed transfer function approach. This method does not require any series truncation and knowledge of the system eigenfunctions as would be required in the modal analysis approach, and is valid for non-self-adjoint systems with inhomogeneous boundary conditions. The approach is suitable for numerical coding and numerical results show that this method generates the same values as the model analysis method. The proposed approach can easily incorporate passive/active control principles for energy harvester design considerations which is the subject of further investigations.

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